On 2-Protected Nodes in Random Digital Trees

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Abstract

In this paper, we consider the number of 2-protected nodes in random digital trees. Results for the mean and variance of this number for tries have been obtained by Gaither, Homma, Sellke and Ward (2012) and Gaither and Ward (2013) and for the mean in digital search trees by Du and Prodinger (2012). In this short note, we show that these previous results and extensions such as the variance in digital search trees and limit laws in both cases can be derived in a systematic way by recent approaches of Fuchs, Hwang and Zacharovas (2010; 2014) and Fuchs and Lee (2014). Interestingly, the results for the moments we obtain by our approach are quite different from the previous ones and contain divergent series which have values by appealing to the theory of Abel summability. We also show that our tools apply to PATRICIA tries, for which the number of 2-protected nodes has not been investigated so far.

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1. Introduction

2-protected nodes in rooted trees are nodes with a distance of at least 2 to every leave (or in other words, nodes which are neither leaves nor parents of leaves). They were introduced by Cheon and Shapiro in [3] as a efficiency measure of organizational schemes. Other applications such as in social networks models have been discussed by Gaither and Ward in [10]. Apart from the practical motiviation, the study of 2-protected nodes and their obvious generalization to k-protected nodes is also interesting from a theoretical point of view. More precisely, k-protected nodes, when considered as a sequence of k, can be interpreted as a profile describing the tree from the fringe to the root. Many other notions of profiles have been investigated in recent years for many different types of random trees. Studying 2-protected nodes constitutes the first step in the study of such a "protected node profile". That is why they have been investigated for many different random trees by many authors: Cheon and Shapiro [3] (random binary trees and random Motzkin trees); Mansour [20] (random k-ary trees); Bóna [2], Devroye and Janson [4], Holmgren and Janson [12, 14], Mahmoud and Ward [18] (random binary search trees); Devroye and Janson [4], Holmgren and Janson [12, 14], Mahmoud and Ward [19] (random recursive trees); Holmgren and Janson [13] (random ternary search trees); and Devroye and Janson [4] (simple generated families of random trees).

In this short note, we are interested in the number of 2-protected nodes in the three main families of random digital trees, namely, random tries (invented by de la Briandais), random PATRICIA tries (invented by Morrison) and random digital search trees (invented by Coffman and Eve). Apart from PATRICIA tries which have not been treated before, results on moments for the number of 2-protected nodes for the other two types of random digital trees already exist. Before recalling these earlier results, we will give the definition of the above three families which are all fundamental data structures in computer science; for more background see [17] or [8, 15].

Now, for the definition, assume that n infinite 0-1 sequences are given which are records to be stored in a binary tree. From these records the trie is a binary tree built as follows: if $n = 1$, the only record is stored in the root; if $n > 1$, then the root is an (empty) internal node and all records are distributed to the two subtrees according to whether their first bit is 0 or 1; finally, the subtrees are built recursively according to the same rules, but by considering subsequent bits. PATRICIA tries are built by the same procedure with the only difference that one-way

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branching is avoided. Finally, digital search trees are also built similarly, but now records may be stored in internal nodes, too. In the sequel, we will equip these tree families with the Bernoulli model which means that bits are i.i.d. Bernoulli random variables with success probability p. We also set $q := 1 - p$ throughout the paper.

We next recall what is known about the number of 2-protected nodes in random digital trees. The first paper which studied this parameter was by Du and Prodinger [5], where an asymptotic expansion of the mean in symmetric digital search trees (i.e., $p = q = 1/2$) was derived. Then, Gaither, Homma, Sellke and Ward [11] proved a similar result for random tries, but for the general case (not only the symmetric case). Moreover, Gaither and Ward in [10] found an asymptotic expansion for the variance of the number of 2-protected nodes in tries and announced a central limit theorem, which was conjectured in their paper. Finally, Devroye and Janson in [4] also announced a (possible) future study of 2-protected nodes in random digital trees based on their method from [4].

The main aim of this paper is to show that all previous results as well as more refined properties for 2-protected nodes in random digital trees can be systematically derived with tools which were developed in a recent series of papers by Fuchs, Hwang and Zacharovas [15, 8] and a paper of Fuchs and Lee [9] (these papers were concerned with general frameworks for studying stochastic properties of parameters such as 2-protected nodes in random digital trees; for details see below). Moreover, our tools also apply straightforwardly to PATRICIA tries, which have not been treated before. For mean and variance, we will compare our results with previous results (if already known). Interestingly, the expressions we obtain will be rather different. In particular, the periodic functions in our expressions for tries will considerably differ from those in [10, 11] and will contain divergent series which can be made convergent by appealing to the theory of Abel summability. This is a new phenomena which has not been present in any of the examples studied in [8]. Apart from considering moments, we will also look at limiting distributions and prove (univariate and bivariate) central limit theorems for the number of 2-protected nodes in the three types of random digital trees. For tries, this will confirm the above mentioned conjecture of [10]. In all other cases, our results are new.

We conclude the introduction with a short sketch of the paper. In the next section, we will recall the framework from [8] and [9]. Moreover, we will sketch a similar framework for symmetric digital search trees which is based on [15] and which was recently obtained by Lee in his Ph.D. thesis [16]. In Section 3, we will discuss our results for random tries and PATRICIA tries. Finally, Section 4 will contain corresponding results for symmetric digital search trees. In the proof of all of our results, we will be deliberately brief since (i) as mentioned above our intention is to show that results for 2-protected nodes for random digital trees follow quite straightforwardly from previous studies and (ii) we do not want to repeat things which have appeared in previous works.

Notations. Throughout the paper, the number of 2-protected nodes in a random digital tree of size n under the Bernoulli model will be denoted by $X_n^{(\star)}$ with $\star \in \{T, P, D\}$, depending on whether tries, PATRICIA tries, or digital search trees are considered. Moreover, for some function $G(x)$, we will use the notation

$$\mathscr{F}[G](x) := \begin{cases} \dfrac{1}{h} \displaystyle\sum_{k \in \mathbb{Z}\setminus\{0\}} G(-1+\chi_k) e^{2k\pi i x}, & \text{if } \frac{\log p}{\log q} \in \mathbb{Q}; \\ 0, & \text{if } \frac{\log p}{\log q} \notin \mathbb{Q}, \end{cases}$$

where $h = -p \log p - q \log q$ and $\chi_k = 2rk\pi i / \log p$ when $\log p / \log q = r/l$ with $\gcd(r, l) = 1$.

2. Preliminaries

In this section, we are going to recall the results from [8] and [9]. We will state them in a form convenient for the applications below. We first need the following notation.

Definition 1. Let $\tilde{f}(z)$ be an entire function and $\alpha, \gamma \in \mathbb{R}$. Then, we say that $\tilde{f}(z)$ is JS-admissible (named after Jacquet and Szpankowski who did important work on these functions) and write $\tilde{f}(z) \in \mathscr{JS}$ (or more precisely, $\tilde{f}(z) \in \mathscr{JS}_{\alpha,\gamma}$) if for $0 < \phi < \pi/2$ and all $|z| \geq 1$ the following two conditions hold.

(I) Uniformly for $|\arg(z)| \leq \phi$,
$$\tilde{f}(z) = \mathcal{O}\left(|z|^{\alpha} (\log_+ |z|)^{\gamma}\right),$$
where $\log_+ x := \log(1 + x)$.

(O) Uniformly for $\phi \leq |\arg(z)| \leq \pi$,

$$f(z) := e^z \tilde{f}(z) = \mathcal{O}\left(e^{(1-\epsilon)|z|}\right),$$

where $\epsilon > 0$.

The papers [8, 9] were concerned with general frameworks for deriving asymptotic expansions of moments and central limit theorems of so-called additive shape parameters in random tries. Here, additive shape parameters are parameters which can be recursively computed as follows: first compute the shape parameter for the two subtrees, add them up and add a cost. From a probabilistic point of view, this translates into a distributional recurrence for the shape parameter X_n in a random trie of size n. More precisely, X_n satisfies for $n \geq 2$:

$$X_n \stackrel{d}{=} X_{I_n} + X^*_{n-I_n} + T_n \tag{1}$$

with initial conditions $X_0 = X_1 = 0$; X_n^* has the same distribution as X_n; X_n, X_n^*, I_n and T_n are independent and I_n is the size of the left subtree which under the Bernoulli model has the distribution

$$P(I_n = k) = \binom{n}{k} p^k q^{n-k}, \qquad (0 \leq k \leq n).$$

Before we can state the results from [8, 9], we need some notation. First, denote the Poisson generating functions of first and second moment of T_n by

$$\tilde{g}_1(z) := e^{-z} \sum_{n \geq 2} \mathbb{E}(T_n) \frac{z^n}{n!}, \qquad \tilde{g}_2(z) := e^{-z} \sum_{n \geq 2} \mathbb{E}(T_n^2) \frac{z^n}{n!}.$$

The use of Poisson generating functions has a long history in the analysis of random digital trees; see [8] or some other of the numerous previous papers on random digital trees (many of which are cited in [8]). Moreover, set

$$\tilde{V}_T(z) := \tilde{g}_2(z) - \tilde{g}_1(z)^2 - z\tilde{g}_1'(z)^2.$$

Then, the main results of [8, 9] combined into one result read as follows.

Theorem 1 ([8, 9]). *Assume that $\tilde{g}_1(z) \in \mathscr{JS}_{\alpha_1,\gamma_1}$ with $\alpha_1 < 1/2$, $\tilde{g}_2(z) \in \mathscr{JS}$ and $\tilde{V}_T(z) \in \mathscr{JS}_{\alpha_2,\gamma_2}$ with $\alpha_2 < 1$. Then,*

$$\mathbb{E}(X_n) \sim \frac{G_1(-1)}{h} n + n\mathscr{F}[G_1](r \log_{1/p} n), \qquad \mathrm{Var}(X_n) \sim \frac{G_2(-1)}{h} n + n\mathscr{F}[G_2](r \log_{1/p} n),$$

where $G_1(\omega), G_2(\omega)$ are computable functions (see Remark 2 below). Moreover, assume in addition that $\|T_n\|_s = o(\sqrt{n})$, $2 < s \leq 3$ and $\mathrm{Var}(X_n) \geq cn$ for n large enough with $c > 0$, then

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathrm{Var}(X_n)}} \xrightarrow{d} N(0, 1).$$

Remark 1. The error terms in the asymptotic expansion of mean and variance are $o(n)$. Improvements beyond this bound are subtle because they involve Diophantine approximation properties of $\log p/\log q$; see Flajolet, Roux and Valleé [7] for details. Note that because of this reason, the error terms in the asymptotic expansions of [10] and [11] are both erroneous.

Remark 2. In [8], the following expressions for $G_1(\omega)$ and $G_2(\omega)$ were given

$$G_1(\omega) = \int_0^\infty \tilde{g}_1(z) z^{\omega-1} \mathrm{d}z, \qquad G_2(\omega) = \int_0^\infty \left(\tilde{V}_T(z) + \tilde{\phi}_1(z) + \tilde{\phi}_2(z)\right) z^{\omega-1} \mathrm{d}z,$$

where

$$\begin{aligned}
\tilde{\phi}_1(z) &:= \tilde{h}(z) - 2\tilde{g}_1(z)\left(\tilde{f}_1(pz) + \tilde{f}_1(qz)\right) - 2z\tilde{g}_1'(z)\left(p\tilde{f}_1'(pz) + q\tilde{f}_1'(qz)\right) \\
\tilde{\phi}_2(z) &:= pqz\left(\tilde{f}_1'(pz) - \tilde{f}_1'(qz)\right)^2.
\end{aligned}$$

Here, $\tilde{f}_1(z)$ denotes the Poisson generating function of $\mathbb{E}(X_n)$ and

$$\tilde{h}(z) := 2e^{-z}\sum_{n\geq 0}\mathbb{E}(T_n)\sum_{0\leq k\leq n}\binom{n}{k}p^k q^{n-k}\left(\mathbb{E}(X_n)+\mathbb{E}(X_{n-k})\right)\frac{z^n}{n!}.$$

Note that both are Mellin integrals; see Flajolet, Gourdon and Dumas [6]. Also, in [8], the authors explained via many examples how to evaluate these two Mellin integrals; see in particular the general discussion on page 18 in [8].

Remark 3. Theorem 1 remains valid in many cases for which in (1) we allow T_n to depend on I_n. The only crucial difference is that $\tilde{h}(z)$ in the previous remark has to be replaced by

$$\tilde{h}(z) := 2e^{-z}\sum_{n\geq 0}\sum_{0\leq k\leq n}\binom{n}{k}p^k q^{n-k}\left(\mathbb{E}(X_n)+\mathbb{E}(X_{n-k})\right)\mathbb{E}(T_n|I_n=k)\frac{z^n}{n!}$$

and one has to show with this new $\tilde{h}(z)$ that $\tilde{\phi}_1(z)\in\mathscr{J\!S}_{\alpha,\gamma}$ with $\alpha<1$. When T_n is given explicitly, this can often be verified by directly computing $\tilde{\phi}_1(z)$ and using closure properties of JS-admissible functions; see Lemma 2.3 in [15].

We next note that a similar result as Theorem 1 can be given for additive shape parameters of PATRICIA tries, too; see [8] for details. Moreover, by combining the approach of [15] with the tools from [8, 9] also additive shape parameters of symmetric digital search trees can be treated similarly. Since the resulting framework has not been stated before in the research literature, we will give some details without proofs; for the proofs see [16].

First, additive shape parameters X_n of random symmetric digital search trees of size n satisfy a similar recurrence to (1), namely, we have for $n\geq 1$:

$$X_{n+1}\stackrel{d}{=}X_{I_n}+X^*_{n-I_n}+T_n$$

with the same assumptions as above except

$$P(I_n=k)=2^{-n}\binom{n}{k},\qquad (0\leq k\leq n)$$

since now only the symmetric case is considered. The main difference is a innocent looking shift on the left hand side of the recurrence. However, this shift changes drastically the underlying analytic problem; for details see [15]. Now, with the same notation as above, we have the following general result for symmetric digital search trees.

Theorem 2. *Assume that $\tilde{g}_1(z)\in\mathscr{J\!S}_{\alpha_1,\gamma_1}$ with $\alpha_1<1/2$, $\tilde{g}_2(z)\in\mathscr{J\!S}$ and $\tilde{V}_T(z)\in\mathscr{J\!S}_{\alpha_2,\gamma_2}$ with $\alpha_2<1$. Then*

$$\mathbb{E}(X_n)\sim\frac{n}{\log 2}\sum_{k\in\mathbb{Z}}\frac{G_1(2+\chi_k)}{\Gamma(2+\chi_k)}n^{\chi_k},\qquad \mathrm{Var}(X_n)\sim\frac{n}{\log 2}\sum_{k\in\mathbb{Z}}\frac{G_2(2+\chi_k)}{\Gamma(2+\chi_k)}n^{\chi_k},$$

where $G_1(\omega),G_2(\omega)$ are computable functions (see Remark 4 below). Moreover, assume in addition that $\|T_n\|_s=o(\sqrt{n})$, $2<s\leq 3$ and $\mathrm{Var}(X_n)\geq cn$ for n large enough with $c>0$, then

$$\frac{X_n-\mathbb{E}(X_n)}{\sqrt{\mathrm{Var}(X_n)}}\stackrel{d}{\longrightarrow}N(0,1).$$

Remark 4. Again, one can make the functions $G_1(\omega)$ and $G_2(\omega)$ entirely explicit. More precisely, we have

$$G_1(\omega)=\int_0^\infty\frac{s^{\omega-1}}{Q(-2s)}\left(\int_0^\infty e^{-sz}\tilde{g}_1(z)\mathrm{d}z\right)\mathrm{d}s,$$
$$G_2(\omega)=\int_0^\infty\frac{s^{\omega-1}}{Q(-2s)}\left(\int_0^\infty e^{-sz}\left(\tilde{V}_T(z)+\tilde{\phi}(z)\right)\mathrm{d}z\right)\mathrm{d}s,$$

where $Q(s)=\prod_{\ell\geq 1}(1-s2^{-\ell})$ and

$$\tilde{\phi}(z)=\tilde{h}(z)-4\tilde{g}_1(z)\tilde{f}_1\left(\frac{z}{2}\right)-2z\tilde{g}_1'(z)\tilde{f}_1'\left(\frac{z}{2}\right)+z\tilde{f}_1''(z)^2.$$

Here, $\tilde{h}(z)$ and $\tilde{f}_1(z)$ are as in Remark 2. Note that these integrals are now the Laplace transform of a Mellin transform. In [15], the evaluation of these integrals was discussed for many shape parameters in symmetric digital search trees.

Remark 5. We note that Remark 3 similarly holds also for Theorem 2 and for the remark above.

3. 2-Protected Nodes in Tries and PATRICIA Tries

2-Protected Nodes in Tries. We start with tries. The main observation is that the number of 2-protected nodes is indeed an additive shape parameter: it can be computed recursively by computing the number for the two subtrees and then adding 1 if the root itself is 2-protected. The latter happens if and only if neither the left nor the right subtree contains only one data. This leads to the following distributional recurrence for $X_n^{(T)}$:

$$X_n^{(T)} \stackrel{d}{=} \begin{cases} X_{n-1}^{(T)}, & \text{if } I_n \in \{1, n-1\}; \\ X_{I_n}^{(T)} + X_{n-I_n}^{(T)*} + 1, & \text{otherwise} \end{cases} \qquad (n \geq 2)$$

with initial conditions $X_0^{(T)} = X_1^{(T)} = 0$ and notation as in the previous section. Thus, $X_n^{(T)}$ satisfies (1) with

$$T_n := \begin{cases} 0, & \text{if } I_n \in \{1, n-1\}; \\ 1, & \text{otherwise.} \end{cases}$$

Consequently, the number of 2-protected nodes in tries falls into the framework of the previous section and a direct application of Theorem 1 gives the following theorem.

Theorem 3 (2-protected nodes in tries)**.** *The mean and variance of the number of 2-protected nodes in tries satisfy*

$$\mathbb{E}(X_n^{(T)}) \sim \frac{G_1^{(T)}(-1)}{h} n + n\mathscr{F}[G_1^{(T)}](r \log_{1/p} n), \qquad \mathrm{Var}(X_n^{(T)}) \sim \frac{G_2^{(T)}(-1)}{h} n + n\mathscr{F}[G_2^{(T)}](r \log_{1/p} n),$$

where $G_1^{(T)}(\omega), G_2^{(T)}(\omega)$ are computable functions (see Proposition 1 below). Moreover, the number of 2-protected nodes in tries satisfies the central limit theorem

$$\frac{X_n^{(T)} - \mathbb{E}(X_n^{(T)})}{\sqrt{\mathrm{Var}(X_n^{(T)})}} \xrightarrow{d} N(0,1).$$

Proof. By a straightforward computation

$$\tilde{g}_1(z) = 1 - e^{-z} + pqz^2 e^{-z} - pze^{-pz} - qze^{-qz}, \qquad \tilde{g}_2(z) = \tilde{g}_1(z) \tag{2}$$

and

$$\begin{aligned} \tilde{V}_T(z) &= \left(1 - e^{-z} + pqz^2 e^{-z} - pze^{-pz} - qze^{-qz}\right)\left(e^{-z} - pqz^2 e^{-z} + pze^{-pz} + qze^{-qz}\right) \\ &\quad - z\left(e^{-z} + 2zpqe^{-z} - pqz^2 e^{-z} - pe^{-pz} - qe^{-z} + p^2 ze^{-pz} + q^2 ze^{-qz}\right)^2. \end{aligned}$$

Thus, by Lemma 2.1 in [15], the required JS-admissibility in Theorem 1 of these functions is checked. Note, however, that this is not enough since $X_n^{(T)}$ and T_n are not independent. Thus, by Remark 3, we also need to consider $\tilde{\phi}_1(z)$ which by another straightforward computations is given as

$$\begin{aligned} \tilde{\phi}_1(z) &= \left(e^{-z} - pqz^2 e^{-z} + pze^{-pz}\right)\tilde{f}_1(pz) + 2\left(e^{-z} - pqz^2 e^{-z} + qze^{-qz}\right)\tilde{f}_1(qz) \\ &\quad - 2z\left(e^{-z} + 2pqze^{-z} - pqz^2 e^{-z} - pe^{-pz} - qe^{-qz} + p^2 ze^{-pz} + q^2 ze^{-qz}\right)\left(p\tilde{f}_1'(pz) + q\tilde{f}_1'(qz)\right). \end{aligned} \tag{3}$$

Also this function is JS-admissible as can be seen from Lemma 2.1 in [15] and Proposition 3.3 in [8]. Thus, the claimed result for the moments follows.

As for the central limit theorem, the only additional assumption which needs some further comments is the linear growth of $\mathrm{Var}(X_n^{(T)})$ for large enough n. This assumption follows from the main result in Schachinger [22]. ▮

As mentioned in the previous result, $G_1^{(T)}(\omega)$ and $G_2^{(T)}(\omega)$ can be computed. For this we use the Remark 2 and the tools from [8] and obtain the following result.

Proposition 1. *(a) For the function $G_1^{(T)}(\omega)$, we have $G_1^{(T)}(\omega) = \Gamma(\omega)\left(-1 + pq\omega(\omega+1) - p^{-\omega}\omega - q^{-\omega}\omega\right)$, where at $\omega = -1$ the function value is understood to be the limit.*

(b) For the function $G_1^{(T)}(\omega)$, with the notation $K_1(\omega) = -1 + pq\omega(\omega+1) - p^{-\omega}\omega - q^{-\omega}\omega$, we have for $G_2^{(T)}(-1)$:

$$\frac{2p^2q}{(1+p)^2} + \frac{2pq^2}{(1+q)^2} - 2pq - \frac{p^2q^2}{4} + 2p\log(1+p) + 2q\log(1+q) + \frac{1}{2} + h - 2\log 2$$
$$+ 2\sum_{\ell\ge 2}(-1)^\ell \frac{K_1(-\ell)}{1-p^\ell-q^\ell}\Big(\left(p^\ell+q^\ell\right)\Big(pq\ell - 3pq - p^{-\ell+2} - q^{-\ell+2} + \frac{1-\ell+p^{-\ell+2}\ell+q^{-\ell+2}\ell}{\ell(\ell-1)}\Big) + \frac{1}{\ell}\Big)$$
$$+ 2\sum_{\ell\ge 1}\frac{(-1)^\ell}{\ell}\frac{p^{1+\ell}+q^{1+\ell}}{1-p^{1+\ell}-q^{1+\ell}}K_1(\ell-1)K_1(-\ell-1) - \frac{1}{h}\left(pq+1-h\right)^2$$
$$-\begin{cases}\dfrac{1}{h\log p}\displaystyle\sum_{j\ge 1}\frac{4rj\pi^2}{\sinh\left(\frac{2rj\pi^2}{\log p}\right)}\Big(\left(\frac{2rj\pi pq}{\log p}\right)^2 + (pq+1)^2\Big), & \textit{if } \frac{\log p}{\log q}\in\mathbb{Q};\\ 0, & \textit{if } \frac{\log p}{\log q}\notin\mathbb{Q}\end{cases}$$

and for $G_2^{(T)}(-1+\chi_k)$ when $\log p/\log q \in \mathbb{Q}$ and $k \neq 0$:

$$-\Gamma(\chi_k+3)2^{-3-\chi_k}p^2q^2 + \Gamma(\chi_k+2)\left(2p^2q(1+p)^{-2-\chi_k} + 2pq^2(1+q)^{-2-\chi_k}\right)$$
$$+ \Gamma(\chi_k+1)\left(-3pq + 2^{-\chi_k}pq - 2^{-1-\chi_k}\right) + \Gamma(\chi_k)\left(1 - 2p(1+p)^{-\chi_k} - 2q(1+q)^{-\chi_k}\right)$$
$$+ \Gamma(\chi_k-1)\left(1-2^{1-\chi_k}\right) + 2\sum_{\ell\ge 2}\frac{(-1)^\ell}{\ell!}\frac{K_1(-\ell)}{1-p^\ell-q^\ell}\Big(pq\left(p^\ell+q^\ell\right)(\ell-1)\Gamma(\chi_k+\ell+1)$$
$$+ \left(1-\ell\left(p^\ell+q^\ell\right)\left(2pq+p^{-\ell+2}+q^{-\ell+2}\right)\right)\Gamma(\chi_k+\ell)$$
$$+ \left(p^\ell+q^\ell\right)\left(1-\ell+p^{-\ell+2}\ell+q^{-\ell+2}\ell\right)\Gamma(\chi_k+\ell-1)\Big)$$
$$+ 2\sum_{\ell\ge 1}\frac{(-1)^\ell}{\ell!}\frac{p^{1+\ell}+q^{1+\ell}}{1-p^{1+\ell}-q^{1+\ell}}K_1(\chi_k+\ell-1)K_1(-\ell-1)\Gamma(\chi_k+\ell)$$
$$- \frac{1}{h}\sum_{j\in\mathbb{Z}}(\chi_j-1)G_1^{(T)}(\chi_j-1)(\chi_{k-j}-1)G_1^{(T)}(\chi_{k-j}-1).$$

Proof. For the proof, we use the formulas from Remark 2. First, for $G_1^{(T)}(\omega)$ plugging (2) into the formula and evaluating the resulting Mellin integral gives part (a). We remark that from inverse Mellin transform, we have the following integral representation

$$\tilde{f}_1(z) = \frac{1}{2\pi i}\int_{(-3/2)}\frac{G_1^{(T)}(s)}{1-p^{-s}-q^{-s}}z^{-s}\mathrm{d}s,$$

where the integration is along the vertical line $\Re(\omega) = -3/2$.

As for part (b), we break the Mellin integral for $G_2^{(T)}(\omega)$ in Remark 2 into three parts according to the three terms in the integrand. The first part is evaluated similar to $G_1^{(T)}(\omega)$ yielding all terms before the first sum in the expressions of part (b) (up to a term which cancels with a term of the last part). For the second part, we use (3) into which we plug the integral expression above. Then, by interchanging integrals, we obtain

$$\int_0^\infty \tilde{\phi}_1(z)z^{\omega-1}\mathrm{d}z = \frac{1}{\pi i}\int_{(-3/2)}\frac{G_1^{(T)}(s)}{1-p^{-s}-q^{-s}}\Big(\left(p^{-s}+q^{-s}\right)\left(\Gamma(\omega-s)-pq\Gamma(\omega-s+2)\right)$$
$$+ p^{-\omega}\Gamma(\omega-s+1) + q^{-\omega}\Gamma(\omega-s+1) + s\left(p^{-s}+q^{-s}\right)\Gamma(\omega-s)\big(1+2pq(\omega-s)$$
$$- pq(\omega-s+1)(\omega-s) + p^{s-\omega+1}(\omega-s-1) + q^{s-\omega+1}(\omega-s-1)\big)\Big). \tag{4}$$

Now, moving the line of integration to $-\infty$ and collecting residues gives the first sum expressions in the expressions of part (b) (note that the resulting sums are divergent. However, if convergence is in the sense of Abel summability, then this step is justified. For a theoretical explanation of why see the remark below). For the final part, we have to consider the Mellin integral

$$pq \int_0^{\infty} \left(\tilde{f}_1'(pz) - \tilde{f}_1'(qz) \right)^2 z^{\omega} \mathrm{d}z. \tag{5}$$

The treatment of this integral is more subtle due to the appearance of the square. In [8], the authors explained a general procedure for handling this term (see page 18 in [8]). Using this procedure, one obtains the last three terms in $G_2^{(T)}(-1)$ and the last two terms in $G_2^{(T)}(-1+\chi_k)$ (up to a term which cancels with a term of the first part). ∎

Remark 6. The expression for $G_1^{(T)}(-1)$ coincides with the one from [11]. Note, however, that no expression for the Fourier coefficients of the periodic function in the case when $\log p/\log q \in \mathbb{Q}$ was given in [11].

Remark 7. Note that our expression for $G_2^{(T)}(-1)$ looks very different from the one in [10] (again no expression for the Fourier coefficients of the periodic function in the case when $\log p/\log q \in \mathbb{Q}$ was given in [10]). In fact, numerical computation indeed reveals a discrepancy. This is explained by the fact that the authors of [10] missed the final term in the above expression of $G_2^{(T)}(-1)$. If this term is added to the expression in [10], then the two expressions are the same as can been shown by a long and tedious computation. For fixed p equality can also be shown numerically, e.g., for the symmetric case, we obtain

$$\frac{G_2^{(T)}(-1)}{\log 2} \approx 0.93443870447019249853\cdots$$

which is the same as the (corrected) value from [10].

Remark 8. In contrast to [10], the series expressions in the above result are not convergent in the classical sense. However, they do converge (and give the correct value) if one uses Abel summability. In order to explain this, we have to justify the step of moving the line of integration to $-\infty$ in (4) (and in similar integrals which appear in the evaluation of (5)).

For the sake of simplicity, we consider the following simplified integral (the argument will be the same for the integrals above)

$$\frac{1}{2\pi i} \int_{(-1/2)} \frac{\Gamma(\omega+1)\Gamma(-\omega)}{1-2^{\omega}} \mathrm{d}\omega. \tag{6}$$

Observe that moving the line of integration to $-\infty$ gives the following divergent series

$$\sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{1-2^{-\ell}}. \tag{7}$$

If on the other hand, we consider Abel summability, then (6) has to be replaced by

$$\frac{1}{2\pi i} \int_{(-1/2)} \frac{\Gamma(\omega+1)\Gamma(-\omega)}{1-2^{\omega}} x^{-\omega} \mathrm{d}\omega$$

with $|x| < 1$. Now, moving the line of integration to $-\infty$ is possibly since x^{ℓ} decays exponentially fast to zero. This yields

$$\sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{1-2^{-\ell}} x^{\ell}.$$

Thus, we have the identity

$$\frac{1}{2\pi i} \int_{(-1/2)} \frac{\Gamma(\omega+1)\Gamma(-\omega)}{1-2^{\omega}} x^{-\omega} \mathrm{d}\omega = \sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{1-2^{-\ell}} x^{\ell}.$$

Now, letting x tend to 1 yields (6) on the left-hand side and the Abel sum of (7) on the right-hand side. This shows our claim.

Note that one alternatively could move the line of integration in (6) to ∞ which gives the convergent series

$$\frac{1}{2}+\sum_{\ell\geq 1}\frac{(-1)^{\ell-1}}{2^{\ell}-1}.$$

This yields the identity

$$\sum_{\ell\geq 1}\frac{(-1)^{\ell-1}}{1-2^{-\ell}}=\frac{1}{2}+\sum_{\ell\geq 1}\frac{(-1)^{\ell-1}}{2^{\ell}-1}$$

which is also (easily) proved directly. Note that the same could also be done in the evaluation of (4) which then gives somehow different final expressions in part (b) of Proposition 1.

2-Protected Nodes in PATRICIA Tries. Now, we turn to PATRICIA tries in which the number of 2-protected nodes is again an additive shape parameter. More precisely, $X_n^{(P)}$ satisfies the following slightly different recurrence:

$$X_n^{(P)} \stackrel{d}{=} \begin{cases} X_n^{(P)}, & \text{if } I_n=\{0,n\};\\ X_{n-1}^{(P)}, & \text{if } I_n=\{1,n-1\};\\ X_{I_n}^{(P)}+X_{n-I_n}^{(T)*}+1, & \text{otherwise}\end{cases}\qquad (n\geq 2)$$

with initial conditions $X_0^{(P)}=X_1^{(P)}=0$ and notation as in Section 2.

Also note that the number of 2-protected nodes in tries and PATRICIA tries are connected as follows via the number N_n of internal nodes in tries

$$X_n^{(P)}=X_n^{(T)}-N_n+n-1. \tag{8}$$

The reason for this is that a PATRICIA trie differs from the trie by the nodes with one-way branching which are counted by N_n-n+1 and all these nodes are 2-protected.

Now, similar to tries, we can apply the tools from [8] to the above recurrence which yields the following result.

Theorem 4 (2-protected nodes in PATRICIA tries)**.** *The mean and variance of the number of 2-protected nodes in PATRICIA tries satisfy*

$$\mathbb{E}(X_n^{(P)})\sim\frac{G_1^{(P)}(-1)}{h}n+n\mathscr{F}[G_1^{(P)}](r\log_{1/p}n),\qquad \mathrm{Var}(X_n^{(P)})\sim\frac{G_2^{(P)}(-1)}{h}n+n\mathscr{F}[G_2^{(P)}](r\log_{1/p}n),$$

where $G_1^{(P)}(\omega), G_2^{(P)}(\omega)$ are computable functions. Moreover, the number of 2-protected nodes in tries satisfies the central limit theorem

$$\frac{X_n^{(P)}-\mathbb{E}(X_n^{(P)})}{\sqrt{\mathrm{Var}(X_n^{(P)})}}\xrightarrow{d}N(0,1).$$

Remark 9. Also, for PATRICIA tries $G_1^{(P)}(\omega), G_2^{(P)}(\omega)$ can be made entirely explicit. The expression are similar as for tries. In order to keep the presentation short, we do not display them here; the interested reader is directed to Yu [23]. We only mention that for the expressions of $G_1^{(P)}(\omega)$ no new computations are necessary since they follow via (8) from the result for tries and the known result for the mean of N_n; see Section 5.1 in [8]. This yields

$$G_1^{(P)}(\omega)=\Gamma(\omega+1)\left(1+pq(\omega+1)-p^{-\omega}-q^{-\omega}\right).$$

This result, e.g., implies that the ratio of 2-protected nodes to all $2n-1$ nodes in symmetric Patricia tries is roughly 18%, whereas the ratio for symmetric tries was roughly 33% and for symmetric digital search trees roughly 31% (see [5] or Section 4 below). Also, note that a similar reasoning as above does not work for the variance since when using (8) one needs the covariance of the size and the number of 2-protected nodes in tries which has not been derived before (it will be, however, given in the next paragraph).

Bivariate Limit Law of N_n and $X_n^{(T)}$. The above two central limit laws for the number of 2-protected nodes in tries and PATRICIA tries can be put under a common umbrella by proving a bivariate central limit theorem for the size and the number of 2-protected nodes in tries. We first need the covariance which by (8) is obtained as

$$\mathrm{Cov}(N_n,X_n^{(T)})=\frac{1}{2}\left(\mathrm{Var}(X_n^{(T)})+\mathrm{Var}(N_n)-\mathrm{Var}(X_n^{(P)})\right).$$

Thus, the above results for the variances of $X_n^{(T)}$ and $X_n^{(P)}$ and the known result for the variance of N_n (e.g., see Section 5.1 in [8]) give the following result.

Proposition 2. *The covariance of the number of internal nodes and the number of 2-protected nodes in tries satisfies*

$$\mathrm{Cov}(N_n, X_n^{(T)}) \sim \frac{H_2(-1)}{h} n + n\mathscr{F}[H_2](r \log_{1/p} n),$$

where

$$H_2(x) = \frac{G_2^{(T)}(x) + G_2^{(N)}(x) - G_2^{(P)}(x)}{2}$$

with $G_2^{(N)}(x)$ is given in Section 5.1 in [8].

Next, we set

$$\Sigma_n = n \begin{pmatrix} G_2^{(N)}(-1)/h + \mathscr{F}[G_2^{(N)}](r \log_{1/p} n) & H_2(-1)/h + \mathscr{F}[H_2](r \log_{1/p} n) \\ H_2(-1)/h + \mathscr{F}[H_2](r \log_{1/p} n) & G_2^{(T)}(-1)/h + \mathscr{F}[G_2^{(T)}](r \log_{1/p} n) \end{pmatrix}.$$

For normalization purpose, we need to show that Σ_n is positive definite for large n.

Lemma 1. *For all n large enough, we have that Σ_n is positive definite.*

Proof. It suffices to show that $\mathrm{Var}(N_n) \geq c_N n$ and $\det \Sigma_n > 0$ for n large enough. The first claim is classical and the second follows with a similar method of proof as Proposition 3 in [9]. ∎

Thus, we can consider $\Sigma_n^{-1/2}$ if n is large enough. Our main result in this section is the following bivariate central limit theorem.

Theorem 5. *The size and the number of 2-protected nodes in tries satisfy the bivariate central limit theorem*

$$\Sigma_n^{-1/2} \begin{pmatrix} N_n - \mathbb{E}(N_n) \\ X_n^{(T)} - \mathbb{E}(X_n^{(T)}) \end{pmatrix} \xrightarrow{d} N(0, I_2),$$

where I_2 denotes the 2×2 unity matrix and $N(0, I_2)$ is the standard two-dimensional normal distribution.

Proof. This follows from our expressions for mean and variance of $X_n^{(T)}$ with a similar method of proof as for Theorem 4 in [9] (which used the multivariate contraction method of Neininger and Rüschendorf [21]). ∎

Note that both the central limit theorem for the number of 2-protected nodes in tries and PATRICIA tries are consequences of this result (the latter follows from (8)).

4. 2-Protected Nodes in Symmetric Digital Search Trees

Here, we consider symmetric digital search trees in which the number of 2-protected nodes X_n is again an additive shape parameter which satisfies:

$$X_{n+1}^{(D)} \stackrel{d}{=} \begin{cases} X_{n-1}^{(D)}, & \text{if } I_n \in \{1, n-1\}; \\ X_{I_n}^{(D)} + X_{n-I_n}^{(D)*} + 1, & \text{otherwise} \end{cases} \qquad (n \geq 0)$$

with initial condition $X_0^{(D)} = 0$ and notation as in Section 2. Thus, applying Theorem 2 in a similar style as for tries gives the following theorem.

Theorem 6 (2-protected nodes in symmetric digital search trees)**.** *The mean and the variance of the number of 2-protected nodes in symmetric digital search trees satisfy*

$$\mathbb{E}(X_n^{(D)}) \sim \frac{n}{\log 2} \sum_{k \in \mathbb{Z}} \frac{G_1^{(D)}(2 + \chi_k)}{\Gamma(2 + \chi_k)}, \qquad \mathrm{Var}(X_n^{(D)}) \sim \frac{n}{\log 2} \sum_{k \in \mathbb{Z}} \frac{G_2^{(D)}(2 + \chi_k)}{\Gamma(2 + \chi_k)},$$

where $G_1^{(D)}(\omega)$ and $G_2^{(D)}(\omega)$ are computable functions. Moreover, the number of 2-protected nodes in symmetric digital search trees satisfies the central limit theorem

$$\frac{X_n^{(D)} - \mathbb{E}(X_n^{(D)})}{\sqrt{\mathrm{Var}(X_n^{(D)})}} \xrightarrow{d} N(0,1).$$

Again, the functions $G_1^{(D)}(\omega)$ and $G_2^{(D)}(\omega)$ can be made explicit by evaluating the expressions in Remark 4. We only state the result for $G_1^{(D)}(\omega)$ since the result for $G_2^{(D)}(\omega)$ is very long.

Proposition 3. *We have,*

$$G_1^{(D)}(\omega) = \kappa(-\omega)\Gamma(\omega)\Gamma(1-\omega) + \frac{Q(2^{\omega-1})}{Q(1)}\Gamma(-\omega)\Gamma(\omega+1),$$

where at $\omega = 2$ the function value is understood to be the limit and

$$\kappa(\omega) = \frac{1}{Q(1)}\sum_{\ell\geq 0} a_{\ell+1}\Bigg(\frac{8\cdot 2^{4\ell} - 32\cdot 2^{3\ell} + 46\cdot 2^{2\ell} - 32\cdot 2^{\ell} + 9}{2^{1-\ell\omega}(2\cdot 2^{\ell}-1)^2(2^{\ell}-1)^3} - \frac{2^{\omega+\ell+3}\left(\omega(2^{\ell+1}-1) + 2^{\ell+1} - 2\right)}{(2\cdot 2^{\ell}-1)^2}$$
$$+ \frac{2^{\ell}\left(2^{\ell}(\omega^2+3\omega-2) - 2^{\ell+1}(\omega^2+4\omega-2) + \omega^2+5\omega+2\right)}{4(2^{\ell}-1)^3}\Bigg)$$

with

$$a_{\ell+1} = \frac{(-1)^{\ell}2^{-\binom{\ell+1}{2}}}{Q_{\ell}}, \qquad Q_{\ell} = \prod_{j=1}^{\ell}(1-2^{-j}).$$

Proof. First, a straightforward computation gives

$$\tilde{g}_1(z) = (z^2/4 - 1)e^{-z} - ze^{-z/2} + 1.$$

Thus, the inner integral of the expression for $G_1^{(D)}(\omega)$ in Remark 4 equals

$$\int_0^{\infty}\tilde{g}_1(z)e^{-sz}\mathrm{d}z = \frac{1}{2(s+1)^3} - \frac{1}{s+1} - \frac{1}{(s+1/2)^2} + \frac{1}{s}.$$

Set

$$\tilde{g}(s) = \frac{1}{2(s+1)^3} - \frac{1}{s+1} - \frac{1}{(s+1/2)^2}.$$

We first evaluate

$$G(\omega) = \int_0^{\infty}\frac{\tilde{g}(s)}{Q(-2s)}s^{\omega-1}\mathrm{d}s.$$

Therefore, we need Equation 2.2.5 of [1] which reads

$$\frac{1}{Q(-2s)} = \sum_{n\geq 0}\frac{(-1)^n}{Q_n}s^n.$$

From this equation and Taylor series expansion

$$\frac{\tilde{g}(s)}{Q(-2s)} = \sum_{r\geq 0}\left(\frac{(r+2)(r+1)}{4} - 1 - (r+1)2^{r+2}\right)(-1)^r s^r\sum_{n\geq 0}\frac{(-1)^n}{Q_n}s_n$$
$$= \sum_{n\geq 0}(-1)^n s^n\sum_{r\geq 0}\frac{1}{Q_{n-r}}\left(\frac{(r+2)(r+1)}{4} - 1 - (r+1)2^{r+2}\right).$$

By the direct mapping theorem from [6] the latter gives

$$G(\omega) \asymp \sum_{n\geq 0}\sum_{r\geq 0}\frac{1}{Q_{n-r}}\left(\frac{(r+2)(r+1)}{4}-1-(r+1)2^{r+2}\right)\frac{(-1)^n}{\omega+n},$$

where the expression on the right hand side of $\asymp$ is the singularity expansion of $G(\omega)$. Next, we use Equation 2.2.6 of [1] which is given by

$$\frac{1}{Q_n}=\frac{1}{Q(1)}\sum_{\ell\geq 0}a_{\ell+1}2^{-n\ell}.$$

This expression yields

$$\begin{aligned}\sum_{r\geq 0}&\frac{1}{Q_{n-r}}\left(\frac{(r+2)(r+1)}{4}-1-(r+1)2^{r+2}\right)\\&=\frac{1}{Q(1)}\sum_{\ell\geq 0}a_{\ell+1}\sum_{r=0}^{n}\left(\frac{(n-r+2)(n-r+1)}{4}2^{-r\ell}-2^{-r\ell}-(n-r+1)2^{n+2-r(\ell+1)}\right)\\&=\frac{1}{Q(1)}\sum_{\ell\geq 0}a_{\ell+1}\left(\frac{8\cdot 2^{4\ell}-32\cdot 2^{3\ell}+46\cdot 2^{2\ell}-32\cdot 2^{\ell}+9}{2^{1-\ell n}(2\cdot 2^{\ell}-1)^2(2^{\ell}-1)^3}-\frac{2^{n+\ell+3}\left(\omega(2^{\ell+1}-1)+2^{\ell+1}-2\right)}{(2\cdot 2^{\ell}-1)^2}\right.\\&\qquad\left.+\frac{2^{\ell}\left(2^{\ell}(n^2+3n-2)-2^{\ell+1}(n^2+4n-2)+n^2+5n+2\right)}{4(2^{\ell}-1)^3}\right).\end{aligned}$$

Note that this is equal to $\kappa(n)$. Thus,

$$G(\omega)\asymp\sum_{n\geq 0}\kappa(n)\frac{(-1)^n}{\omega+n}.$$

Now, with the same argument as in Example 5 in [6], we obtain

$$G(\omega)=\kappa(-\omega)\Gamma(\omega)\Gamma(1-\omega).$$

Moreover, from (28) in [15],

$$\int_0^{\infty}\frac{s^{\omega-2}}{Q(-2s)}\mathrm{d}s=\frac{Q(2^{-\omega})}{Q(1)}\Gamma(-\omega)\Gamma(\omega+1).$$

Collecting everything yields the desired result. ▮

Remark 10. Note that the expression we obtained is different from the one in [5], where only the expression for $G_1^{(D)}(2)/\log 2$ was given. However, they coincide. Numerically,

$$\frac{G_1^{(D)}(2)}{\log 2}\approx 0.30707981393605921828\cdots$$

which is the same as the value obtained in [5].

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