The Subtree Size Profile of Plane-oriented Recursive Trees

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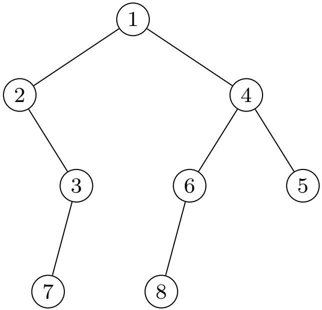
Profiles of Trees

Rooted tree of size n.



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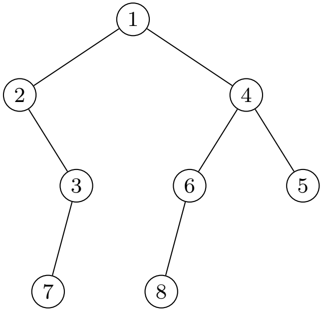

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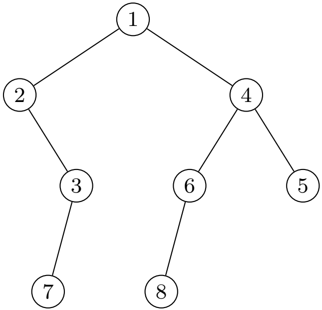

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Subtree size profile:

$\#$ of subtrees of size k.

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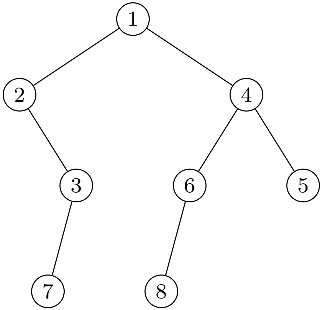

Node profile:

$\#$ of nodes at level k.

Subtree size profile:

$\#$ of subtrees of size k.

Both are a double-indexed sequence $X_{n,k}$.

Recent Studies of Profile

Node Profile: extensively studied for many classes of trees.

Drmota and Gittenberger (1997); Chauvin, Drmota, Jabbour-Hattab (2001); Chauvin, Klein, Marckert, Rouault (2005); Drmota and Hwang (2005); Fuchs, Hwang, Neininger (2006); Hwang (2007); Drmota, Janson, Neininger (2008); Park, Hwang, Nicodeme, Szpankowski (2009); Drmota and Szpankowski (2010); etc.

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Subtree Size Profile: mainly studied for binary trees and increasing trees.

Feng, Miao, Su (2006); Feng, Mahmoud, Su (2007); Feng, Mahmoud, Panholzer (2008); Fuchs (2008); Chang and Fuchs (2010); Dennert and Grübel (2010); etc.

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- Contains information about occurrences of patterns.

  Important in many fields such as Computer Science (compressing, etc.), Mathematical Biology (phylogenetics), etc.

Random Binary Trees

Randomly pick an external node and replace it by a cherry.

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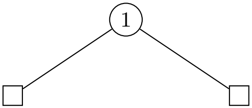

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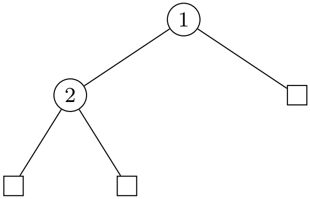

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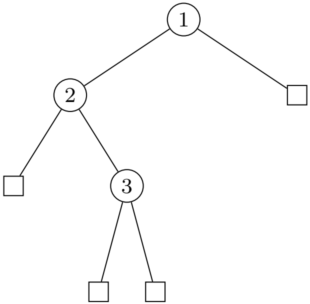

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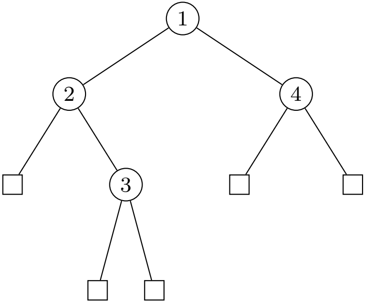

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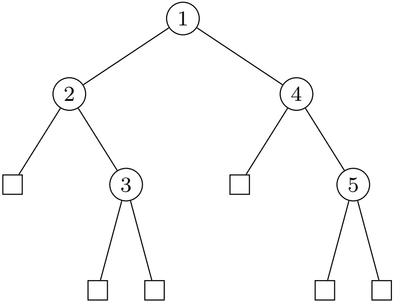

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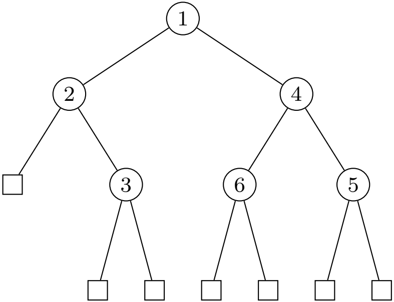

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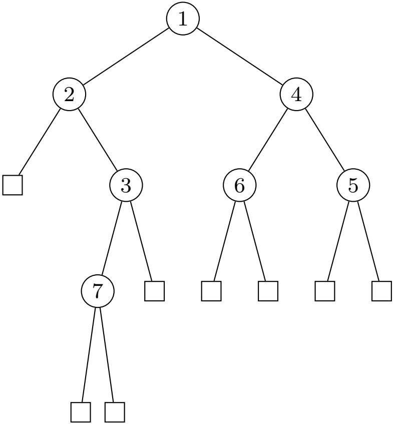

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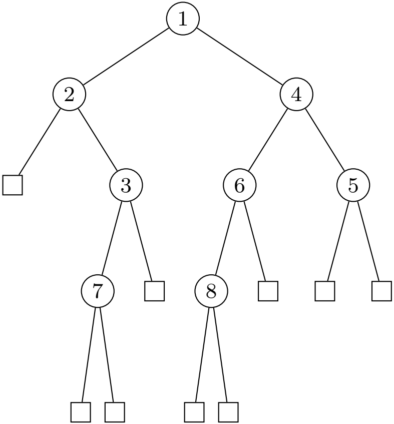

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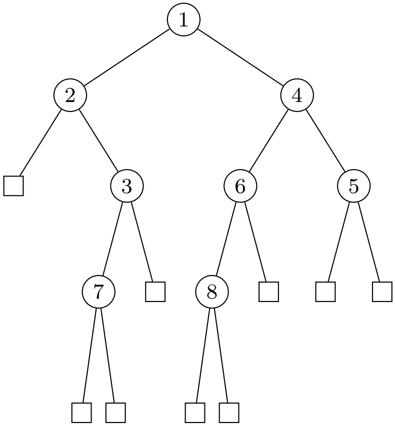

Randomly pick an external node and replace it by a cherry.

Equivalent to random binary search tree.

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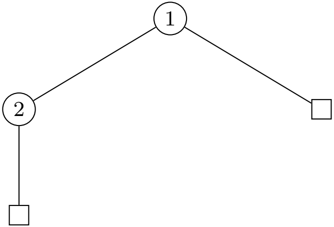

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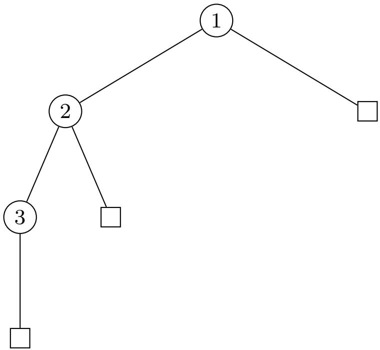

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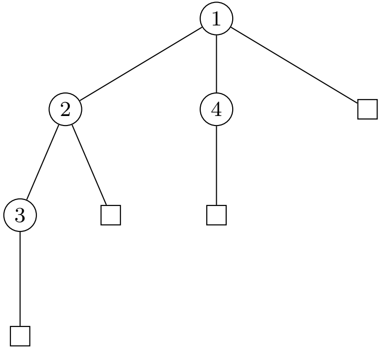

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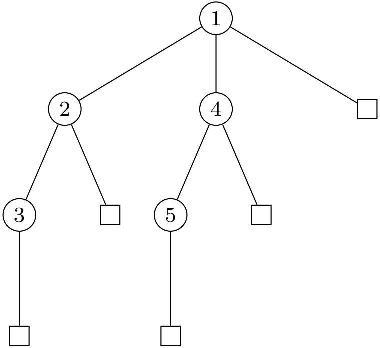

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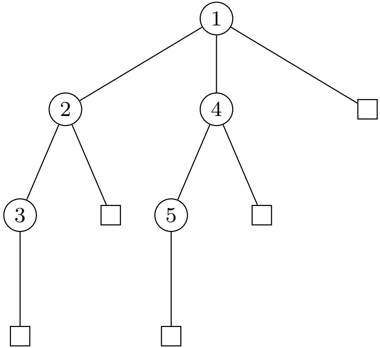

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Same as uniform model for non-plane trees with increasing labels.

Random Plane-oriented Recursive Trees (PORTs)

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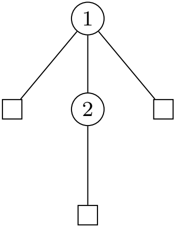

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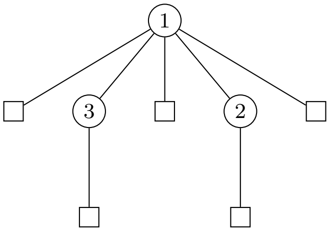

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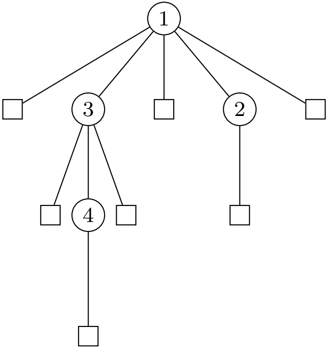

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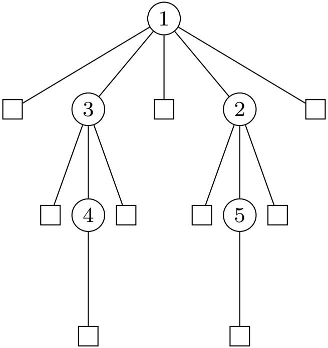

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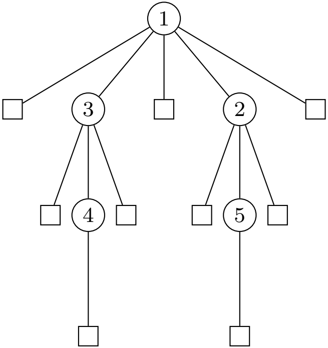

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Same as uniform model for plane trees with increasing labels.

Importance of the Random Models

- **Random Binary Trees**

  Binary search tree, Quicksort, Yule-Harding Model in Phylogenetics, Coalescent Model, etc.

- **Random Recursive Trees**

  Simple model for spread of epidemics, for pyramid schemes, for stemma construction of philology, etc. Also, used in computational geometry and in Hopf algebras.

- **Random PORTs**

  One of the simplest network models (for instance for WWW).

Limit Laws for Random Binary Trees

Theorem (Feng, Mahmoud, Panholzer; F.)

(i) *(Normal range) Let* $k = o\left(\sqrt{n}\right)$*. Then,*

$$\frac{X_{n,k} - \mu_{n,k}}{\sigma_{n,k}} \xrightarrow{d} \mathcal{N}(0,1).$$

(ii) *(Poisson range) Let* $k \sim c\sqrt{n}$*. Then,*

$$X_{n,k} \xrightarrow{d} \mathrm{Poisson}(2c^{-2}).$$

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Similar result holds for random recursive trees as well.

Consequences for Occurrences of Pattern Sizes

Pattern=Subtree on the fringe of the tree.

# of patterns of size k:

$$C_k = \frac{1}{k+1}\binom{2k}{k} \sim \frac{4^k}{\sqrt{\pi}k^{3/2}}.$$

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On the other hand, our result shows:

- Pattern sizes occur until $o(\sqrt{n})$.
- Pattern sizes sporadically exist around $\sqrt{n}$.
- Patterns with sizes beyond $\sqrt{n}$ are unlikely.

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Recall that

$$T_n = \sum_{k=0}^{n-1} k X_{n,k}$$

Hence, if all pattern sizes up to k_0 exist, then

$$\Theta(k_0^2) = \sum_{k \le k_0} k \le T_n = \Theta(n \log n).$$

Thus, pattern sizes beyond $\sqrt{n \log n}$ are very unlikely.

Method of Moments

Theorem

Z_n, Z random variables. If

$$\mathbb{E}(Z_n^m) \longrightarrow \mathbb{E}(Z^m)$$

for all $m \geq 1$ and Z is uniquely determined by its moments, then

$$Z_n \xrightarrow{d} Z.$$

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If Z is standard normal, then

$$\mathbb{E}(Z_n^m) = \begin{cases} 0, & \text{if } m \text{ is odd;} \\ m!/(2^{m/2}(m/2)!), & \text{if } m \text{ is even.} \end{cases}$$

Proof of the Limit Laws

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- Derived a recurrence for centered moments.
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Our approach is easier and can be applied to other random trees.

Theorem (F.)

(i) *(Normal range) Let* $k = o(\sqrt{n})$. *Then,*

$$\frac{X_{n,k} - \mu n}{\sigma\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1),$$

where $\mu = 1/(2k^2)$ *and*

$$\sigma^2 = \frac{8k^2 - 4k - 8}{(4k^2 - 1)^2} - \frac{(2k-3)!!^2}{(k-1)!^2 4^{k-1} k(2k+1)}.$$

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Proof for Fixed k

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where $\bar{B}_k^{[m]}(z)$ is a function of $\bar{A}_k^{[i]}(z)$ with $i < m$.

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Asymptotics of centered moments via induction ("moment-pumping").

Singularity Analysis

Consider

$$\Delta = \{z \ : \ |z| < r, \ z \neq 1/2, \ |\arg(z-1/2)| > \varphi\},$$

where $r > 1$ and $0 < \varphi < \pi/2$.

Theorem (Flajolet and Odlyzko)

(i) *For* $\alpha \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$

$$[z^n](1-2z)^{-\alpha} \sim \frac{2^n n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \frac{\alpha(\alpha-1)}{2n} + \cdots\right).$$

(ii) *Let* $f(z)$ *be analytic in* Δ. *Then,*

$$f(z) = \mathcal{O}\left((1-2z)^{-\alpha}\right) \quad \Rightarrow \quad [z^n]f(z) = \mathcal{O}\left(2^n n^{\alpha-1}\right).$$

Closure Properties

Theorem

Let $f(z)$ be analytic in Δ with

$$f(z)=\sum_{j=0}^{J}c_j(1-2z)^{\alpha_j}+\mathcal{O}\left((1-2z)^{A}\right),$$

where $\alpha_j, A\neq 1$. Then, $\int_0^z f(t)\mathrm{d}t$ is analytic in Δ and

(i) *if $A>-1$, then for some explicit c*

$$\int_0^z f(t)\mathrm{d}t=-\frac{1}{2}\sum_{j=0}^{J}\frac{c_j}{\alpha_j+1}(1-2z)^{\alpha_j+1}+c+\mathcal{O}\left((1-2z)^{A+1}\right);$$

(ii) *if $A<-1$, then as above but without c.*

Asymptotic Expansions

Proposition

$\bar{A}_k^{[m]}(z)$ *is analytic in* Δ.

Moreover,

$$\bar{A}_k^{[2m-1]}(z) = \mathcal{O}\left((1-2z)^{3/2-m}\right)$$

and

$$\bar{A}_k^{[2m]}(z) = \frac{(2m)!(2m-3)!!\sigma^{2m}}{4^m m!}(1-2z)^{1/2-m} + \mathcal{O}\left((1-2z)^{1-m}\right).$$

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From this, by singularity analysis,

$$\mathbb{E}(X_{n,k}-\mu n)^m = \begin{cases} 0, & \text{if } m \text{ is odd;} \\ m!/(2^{m/2}(m/2)!), & \text{if } m \text{ is even.} \end{cases}$$

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$$\bar{A}^{[m]}_{n,k} = \sum_{1 \le j < n} \pi_{n,j}(\bar{A}^{[m]}_{j,k} + \bar{A}^{[m]}_{n-j,k}) + \bar{B}^{[m]}_{n,k},$$

where $\bar{B}^{[m]}_{n,k}$ is a function of $\bar{A}^{[i]}_{n,k}$ with $i < m$ and

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We have

$$\bar{A}_{n,k}^{[m]} = \sum_{k+1 \le j \le n} \frac{(n+1-j)C_j}{C_n} \bar{B}_{j,k}^{[m]}.$$

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We have,

$$\bar{A}_{n,k}^{[2m-1]} = o\left(\left(\frac{n}{k^2}\right)^{m-1/2}\right), \qquad \bar{A}_{n,k}^{[2m]} \sim g_m \left(\frac{n}{2k^2}\right)^m,$$

where

$$g_m = (2m)!/(2^m m!).$$

Asymptotic Expansions

Proposition

Uniformly in n, k, m

$$\bar{A}_{n,k}^{[m]} = \mathcal{O}\left(\max\left\{\frac{n}{k^2}, \left(\frac{n}{k^2}\right)^{m/2}\right\}\right).$$

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Simple Classes of Increasing Trees (i)

Consider rooted, plane trees with increasing labels.

Let ϕ_r be a *weight sequence* with $\phi_0 > 0$ and $\phi_r > 0$ for some $r \geq 2$. Denote by $\phi(\omega)$ the OGF of ϕ_r.

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Consider rooted, plane trees with increasing labels.

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Set

$$\tau_n = \sum_{\#T=n} \omega(T)$$

which is the cumulative weight of all trees of size n.

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Define the probability of a tree of size n as

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Previous Models

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All these models can be obtained from a tree evolution process.

Grown Simple Classes of Increasing Trees

Theorem (Panholzer and Prodinger)

All simple classes of increasing tree which can be obtained via a tree evolution process are

- *Random d-ary trees:* $\phi(\omega) = \phi_0(1 + ct/\phi_0)^d$ *with* $c > 0, d \in \{2, 3, \dots\}$*;*
- *Random increasing trees:* $\phi(\omega) = \phi_0 e^{ct/\phi_0}$ *with* $c > 0$*;*
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For stochastic properties, it is sufficient to consider the cases with $\phi_0 = c = 1$.

Mean for Grown Simple Classes of Increasing Trees

Proposition

For random d-ary trees,

$$\mu_{n,k} := \mathbb{E}(X_{n,k}) = \frac{d((d-1)n+1)}{((d-1)k+d)((d-1)k+1)}.$$

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For generalized random PORTs,

$$\mu_{n,k} := \mathbb{E}(X_{n,k}) = \frac{(r-1)(rn-1)}{(rk+r-1)(rk-1)}.$$

Limit Laws for d-ary Trees

Theorem (F.)

(i) *(Normal range) Let* $k = o(\sqrt{n})$ *and* $k \to \infty$. *Then,*

$$\frac{X_{n,k} - \mu_{n,k}}{\sqrt{\mu_{n,k}}} \xrightarrow{d} \mathcal{N}(0,1).$$

(ii) *(Poisson range) Let* $k \sim c\sqrt{n}$. *Then,*

$$X_{n,k} \xrightarrow{d} \mathrm{Poisson}(2c^{-2}).$$

(iii) *(Degenerate range) Let* $k < n$ *and* $\sqrt{n} = o(k)$. *Then,*

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Limit Laws for Generalized Random PORTs

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- Approach can be refined to obtain Berry-Esseen bounds, LLT and Poisson approximation results.