Metric Diophantine Approximation for Formal Laurent Series over Finite Fields

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Notation

- Field of formal Laurent series:
  $$\mathbb{F}_q((T^{-1})) = \{ f = a_n T^n + a_{n-1} T^{n-1} + \cdots : a_j \in \mathbb{F}_q, a_n \neq 0 \} \cup \{0\}.$$ 

- Valuation induced by the general degree function:
  $$|f| = q^n, \quad |0| = 0.$$ 

- Analogue of $[0, 1)$:
  $$\mathbb{L} = \{ f \in \mathbb{F}_q((T^{-1})) : |f| < 1 \}.$$ 

- Restricting $|\cdot|$ to $\mathbb{L}$ gives compact topological group. Denote by $m$ the unique, translation-invariant (Haar) probability measure.
Approximation Problem - Coprime Solutions

For \( f \in \mathbb{L} \) consider:

\[
\left| f - \frac{P}{Q} \right| < \frac{1}{q^{n+l_n}}, \quad \text{deg } Q = n, \quad Q \text{ monic,} \quad (AP)
\]

where

- \( P, Q \in \mathbb{F}_q[T] \), \( Q \neq 0 \);
- \( l_n \) is a sequence of non-negative integers.
Approximation Problem - Coprime Solutions

For $f \in \mathbb{L}$ consider:

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{n+l_n}}, \text{ deg } Q = n, \ Q \text{ monic},$$

where

- $P, Q \in \mathbb{F}_q[T], Q \neq 0$;
- $l_n$ is a sequence of non-negative integers.

Question:

For a “typical” $f$ (with respect to $m$), what can be said about the number of pairs $(P, Q)$ with $\gcd(P, Q) = 1$ solving the above Diophantine inequality?
Two Results of Inoue & Nakada

Theorem (Inoue & Nakada; 2003)

*AP has either finitely or infinitely many coprime solutions for almost all f.* The latter holds iff

\[ \sum_{n} q^{n-l_n} = \infty. \]
Two Results of Inoue & Nakada

Theorem (Inoue & Nakada; 2003)

AP has either finitely or infinitely many coprime solutions for almost all \( f \). The latter holds iff

\[
\sum_n q^{n-l_n} = \infty.
\]

Theorem (Inoue & Nakada; 2003)

Let \( l_n \geq n \). Then, the number of coprime solutions of AP with \( n \leq N \) satisfies

\[
(1 - q^{-1})\Psi(N) + \mathcal{O} \left( \Psi(N)^{1/2} (\log \Psi(N))^{3/2+\epsilon} \right) \quad \text{a.s.,}
\]

where \( \Psi(N) = \sum_{n \leq N} q^{n-l_n} \).
Some Notation

Assume that

$$\sum_{n} q^{n-l_n} = \infty \quad \text{and} \quad l_n \text{ increasing}.$$ 

Note that the latter implies that $l_n \geq n$ and $l_n - n$ is non-decreasing.
Some Notation

Assume that

$$\sum_n q^{n-l_n} = \infty \quad \text{and} \quad l_n \text{ increasing.}$$

Note that the latter implies that $l_n \geq n$ and $l_n - n$ is non-decreasing.

Define

$$F(N) = \begin{cases} q^{-2l-2} (q^{l+1}(q - 1) - (2l + 1)(q - 1)^2) N, & \text{if } l_n - n \to l; \\ (1 - q^{-1})\Psi(N), & \text{if } l_n - n \to \infty, \end{cases}$$

where $\Psi(N) = \sum_{n \leq N} q^{n-l_n}$.
Assume that
\[ \sum_n q^{n-l_n} = \infty \quad \text{and} \quad l_n \text{ increasing.} \]

Note that the latter implies that \( l_n \geq n \) and \( l_n - n \) is non-decreasing.

Define
\[
F(N) = \begin{cases} 
q^{-2l-2} \left( q^{l+1} (q - 1) - (2l + 1)(q - 1)^2 \right) N, & \text{if } l_n - n \to l; \\
(1 - q^{-1}) \Psi(N), & \text{if } l_n - n \to \infty,
\end{cases}
\]

where \( \Psi(N) = \sum_{n \leq N} q^{n-l_n} \).

Finally set
\[
Z_N(f) = \# \text{ coprime solutions of AP with } n \leq N.
\]
Theorem (Deligero & Nakada; 2004)

As \( N \to \infty \),

\[
\frac{Z_N - (1 - q^{-1})\Psi(N)}{\sqrt{F(N)}} \overset{d}{\to} \mathcal{N}(0, 1),
\]

where \( \Psi(N) = \sum_{n \leq N} q^{n-l_n} \).
Theorem (Deligero & Nakada; 2004)

As $N \to \infty$,

$$
\frac{Z_N - (1 - q^{-1})\Psi(N)}{\sqrt{F(N)}} \xrightarrow{d} N(0, 1),
$$

where $\Psi(N) = \sum_{n \leq N} q^{n-l_n}$.

Theorem (Deligero, F., Nakada; 2007)

We have,

$$
\limsup_{N \to \infty} \frac{|Z_N(f) - (1 - q^{-1})\Psi(N)|}{\sqrt{2F(N) \log \log F(N)}} = 1 \quad \text{a.s.},
$$

where $\Psi(N) = \sum_{n \leq N} q^{n-l_n}$. 
For $f \in \mathbb{L}$ consider:

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{n+l_n}}, \text{ deg } Q = n, \ Q \text{ monic}, \quad \text{(AP)}$$

where

- $P, Q \in \mathbb{F}_q[T], Q \neq 0$;
- $l_n$ is a sequence of non-negative integers.

**Question:**

For a “typical” $f$ (with respect to $m$), what can be said about the number of pairs $(P, Q)$ solving the above Diophantine inequality?
Theorem (Nakada & Natsui; 2006)

Assume that

(i) $l_n$ is increasing, $\sum_n q^{n-l_n} = \infty$;
(ii) The sequence recursively defined by

$$j_1 = \min\{n \geq 2 : l_n - l_{n-1} > 1\};$$
$$j_k = \min\{n > j_{k-1} : l_n - l_{n-1} > 1\}$$

is lacunary.

Then, the number of solutions of AP with $n \leq N$ is asymptotic to

$$\Psi(N) = \sum_{n \leq N} q^{n-l_n}.$$
An improved Result

Theorem (F.)

Let $l_n \geq n$. Then, the number of solutions of AP with $n \leq N$ satisfies

$$
\Psi(N) + O \left( \Psi(N)^{1/2} (\log \Psi(N))^{2+\epsilon} \right) \quad \text{a.s.,}
$$

where

$$
\Psi(N) = \sum_{n \leq N} q^{n-l_n}.
$$
Inhomogeneous Diophantine Approximation

For \( f, g \in \mathbb{L} \) consider:

\[
|Qf - g - P| < \frac{1}{q^{l_n}}, \quad \text{deg } Q = n, \quad Q \text{ monic,} \tag{IAP}
\]

where \( P, Q \) and \( l_n \) are as before.
Inhomogeneous Diophantine Approximation

For $f, g \in \mathbb{L}$ consider:

$$|Qf - g - P| < \frac{1}{q^{l_n}}, \ \text{deg} \ Q = n, \ Q \ \text{monic},$$

(IAP)

where $P, Q$ and $l_n$ are as before.

Different cases:

(D) **Double metric case**: both $f, g$ random;
Inhomogeneous Diophantine Approximation

For $f, g \in \mathbb{L}$ consider:

$$|Qf - g - P| < \frac{1}{q^{l_n}}, \; \text{deg} \; Q = n, \; Q \text{ monic},$$

(IAP)

where $P, Q$ and $l_n$ are as before.

Different cases:

(D) Double metric case: both $f, g$ random;

(S) Single metric cases:

(S1) $g$ fixed, $f$ random;

(S2) $f$ fixed, $g$ random.
Double Metric Case

Theorem (Ma & Su; 2008)

IAP for \((D)\) has either finitely or infinitely many solutions for almost all \((f, g)\). The latter holds iff

\[
\sum_{n} q^{n-l_n} = \infty.
\]
Double Metric Case

Theorem (Ma & Su; 2008)

IAP for (D) has either finitely or infinitely many solutions for almost all \((f, g)\). The latter holds iff

\[
\sum_{n} q^{n-l_n} = \infty.
\]

Theorem (F.)

Let \(l_n \geq n\). Then, the number of solutions of IAP for (D) with \(n \leq N\) satisfies

\[
\Psi(N) + \mathcal{O} \left( \Psi(N)^{1/2} \left( \log \Psi(N) \right)^{3/2+\varepsilon} \right)
\]
a.s.,

where \(\Psi(N) = \sum_{n \leq N} q^{n-l_n}\).
Single Metric Cases

Theorem (F.)

Let $l_n \geq n$. Then, the number of all solutions of IAP for (S1) with $n \leq N$ satisfies

$$\Psi(N) + O\left(\Psi(N)^{1/2} (\log \Psi(N))^{2+\epsilon}\right),$$

where

$$\Psi(N) = \sum_{n \leq N} q^{n-l_n}.$$
Single Metric Cases

Theorem (F.)

Let \( l_n \geq n \). Then, the number of all solutions of IAP for (S1) with \( n \leq N \) satisfies

\[
\Psi(N) + \mathcal{O} \left( \Psi(N)^{1/2} (\log \Psi(N))^{2+\epsilon} \right),
\]

where

\[
\Psi(N) = \sum_{n \leq N} q^{n-l_n}.
\]

Theorem (F.)

A similar result for (S2) cannot hold.

More precisely, for any \( l_n \) there exists an \( f \) such that the number of solutions of (S2) is finite almost surely.
Restricted Diophantine Approximation

For $f, g \in \mathbb{L}$ consider:

$$|F(Q)f - g - P| < \frac{1}{q^{l_n}}, \deg Q = n, \ Q \text{ monic},$$

(RAP)

where $P, Q, l_n$ are as before and $F$ is a map from $\mathbb{F}_q[T]$ to $\mathbb{F}_q[T]$. 
Restricted Diophantine Approximation

For \( f, g \in \mathbb{L} \) consider:

\[
|F(Q)f - g - P| < \frac{1}{q^{l_n}}, \quad \deg Q = n, \quad Q \text{ monic},
\]

(RAP)

where \( P, Q, l_n \) are as before and \( F \) is a map from \( \mathbb{F}_q[T] \) to \( \mathbb{F}_q[T] \).

Assumption and Notation:

- \( \deg Q \leq \deg Q' \quad \Rightarrow \quad F(Q) \leq F(Q') \);
Restricted Diophantine Approximation

For \( f, g \in \mathbb{L} \) consider:

\[
|F(Q)f - g - P| < \frac{1}{q^{l_n}}, \quad \text{deg} \ Q = n, \ Q \text{ monic}, \tag{RAP}
\]

where \( P, Q, l_n \) are as before and \( F \) is a map from \( \mathbb{F}_q[T] \) to \( \mathbb{F}_q[T] \).

Assumption and Notation:

- \( \text{deg} \ Q \leq \text{deg} \ Q' \quad \Rightarrow \quad F(Q) \leq F(Q') \);
- Set

\[
\mathcal{F} = \{ Q : \ Q \text{ monic and } F(Q) \neq 0 \}.
\]

and

\[
\mathcal{F}_n = \{ Q : Q \in \mathcal{F}, \ \text{deg} \ Q = n \}.
\]
A Theorem for Special $F$

Theorem (F.)

Let $l_n \geq n$ and assume that $F(Q) \in \{Q, 0\}$. Then, the number of solutions of RAP with $Q \in F$ and $n \leq N$ satisfies

$$
\Psi(N, F) + \mathcal{O}\left(\Psi(N)^{1/2} (\log \Psi(N))^{2+\epsilon}\right) \quad \text{a.s.,}
$$

where

$$
\Psi(N) = \sum_{n \leq N} q^{n-l_n}, \quad \Psi(N, F) = \sum_{n \leq N} \#F_n q^{-l_n}.
$$

Remark: This gives a meaningful formula whenever $\lim \inf_{n \to \infty} \#F_n q^{-l_n} > 0$. 
A Theorem for Special $F$

**Theorem (F.)**

Let $l_n \geq n$ and assume that $F(Q) \in \{Q, 0\}$. Then, the number of solutions of RAP with $Q \in F$ and $n \leq N$ satisfies

$$\Psi(N, F) + O \left( \Psi(N)^{1/2} \left( \log \Psi(N) \right)^{2+\epsilon} \right) \text{ a.s.,}$$

where

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**Remark:**

This gives a meaningful formula whenever

$$\liminf_{n \to \infty} \#F_n q^{-n} > 0.$$
Consequences

Corollary

Let \( l_n \geq n \) and set \( \Psi(N) = \sum_{n \leq N} q^{n-l_n} \).

(i) Let \( C, D \in \mathbb{F}_q[T] \) with \( \deg C < \deg D \). Then, the number of solutions of IAP with \( Q \equiv C \cdot D \) and \( n \leq N \) satisfies

\[
\frac{1}{|D|} \Psi(N) + \mathcal{O} \left( (\Psi(N))^{1/2} (\log \Psi(N))^{2+\epsilon} \right) \quad \text{a.s.}
\]
Consequences

Corollary

Let \( l_n \geq n \) and set \( \Psi(N) = \sum_{n \leq N} q^{n-l_n} \).

(i) Let \( C, D \in \mathbb{F}_q[T] \) with \( \deg C < \deg D \). Then, the number of solutions of IAP with \( Q \equiv C(D) \) and \( n \leq N \) satisfies

\[
\frac{1}{|D|} \Psi(N) + O \left( (\Psi(N))^{1/2} (\log \Psi(N))^{2+\epsilon} \right) \quad \text{a.s.}
\]

(ii) The number of solutions of IAP with \( Q \) square-free and \( n \leq N \) satisfies

\[
(1 - q^{-1}) \Psi(N) + O \left( (\Psi(N))^{1/2} (\log \Psi(N))^{2+\epsilon} \right) \quad \text{a.s.}
\]
A Theorem for General $F$

Theorem (F.)

Let $l_n \geq n$. Then, the number of solutions of RAP with $Q \in \mathcal{F}$ and $n \leq N$ satisfies

$$\Psi(N, \mathcal{F}) + O\left(\Psi_0(N)^{1/2} (\log \Psi_0(N))^{3/2+\epsilon}\right) \quad \text{a.s.,}$$

where

$$\Psi(N, \mathcal{F}) = \sum_{n \leq N} \#\mathcal{F}_n q^{-l_n}$$

and

$$\Psi_0(N) = \sum_{n \leq N} q^{-l_n} \sum_{m \leq n} \sum_{Q \in \mathcal{F}_n} \sum_{Q' \in \mathcal{F}_m} \left\lfloor \frac{|\gcd(F(Q), F(Q'))|}{|F(Q)|} \right\rfloor$$
Consequences

Corollary

Let $l_n \geq n$.

(i) The number of solutions of IAP with $Q$ irreducible and $n \leq N$ satisfies

$$
\Psi_1(N) + O\left(\Psi_1(N)^{1/2} (\log \Psi_1(N))^{3/2+\epsilon}\right) \quad \text{a.s.,}
$$

where $\Psi_1(N) = \sum_{n \leq N} n^{-1} q^{n-l_n}$.
Consequences

Corollary

Let $l_n \geq n$.

(i) The number of solutions of IAP with $Q$ irreducible and $n \leq N$ satisfies

$$\Psi_1(N) + \mathcal{O}\left(\Psi_1(N)^{1/2} (\log \Psi_1(N))^{3/2+\epsilon}\right) \quad \text{a.s.},$$

where $\Psi_1(N) = \sum_{n \leq N} n^{-1} q^{n-l_n}$.

(ii) Let $F(Q) = Q^t$ with $t \geq 2$. Then, the number of solutions of RAP with $n \leq N$ satisfies

$$\Psi(N) + \mathcal{O}\left(\Psi(N)^{1/2} (\log \Psi(N))^{3/2+\epsilon}\right) \quad \text{a.s.},$$

where $\Psi(N) = \sum_{n \leq N} q^{n-l_n}$. 
Simultaneous Diophantine approximation

For \((f_1, \ldots, f_d) \in \mathbb{L} \times \cdots \times \mathbb{L}\) consider:

\[
\left| f_j - \frac{P_j}{Q} \right| < \frac{1}{q^{n+l^{(j)}}}, \quad 1 \leq j \leq d, \quad \text{deg } Q = n, \quad Q \text{ monic,}
\]

where \(P_j, Q\) and \(l^{(j)}_n\) are as before. Set \(l_n = \sum_j l^{(j)}_n\).
Simultaneous Diophantine approximation

For \((f_1, \ldots, f_d) \in \mathbb{L} \times \cdots \times \mathbb{L}\) consider:

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\left| f_j - \frac{P_j}{Q} \right| < \frac{1}{q^{n+l_n(j)}}, \quad 1 \leq j \leq d, \quad \text{deg } Q = n, \quad Q \text{ monic},
\]

(SAP)

where \(P_j, Q\) and \(l_n(j)\) are as before. Set \(l_n = \sum_j l_n(j)\).

**Theorem (F.)**

Let \(l_n \geq n\). Then, the number of all solutions of AP with \(n \leq N\) satisfies

\[
\Psi(N) + \mathcal{O}\left(\Psi(N)^{1/2} \left(\log \Psi(N)\right)^{2+\epsilon}\right),
\]

where

\[
\Psi(N) = \sum_{n \leq N} q^{n-l_n}.
\]
Theorem (F.)

SAP has either finitely or infinitely many coprime solutions for almost all \( f \). The latter holds iff

\[
\sum_{n} q^{n-l_n} = \infty.
\]
Back to the Case of Coprime Solutions

Theorem (F.)

SAP has either finitely or infinitely many coprime solutions for almost all $f$. The latter holds iff

$$\sum_{n} q^{n-l_n} = \infty.$$ 

Theorem (F.)

Let $l_n \geq n$. Then, the number of coprime solutions of SAP with $n \leq N$ satisfies

$$c_0 \Psi(N) + O \left( \Psi(N)^{1/2+\epsilon} \right) \quad a.s.,$$

where $\Psi(N) = \sum_{n \leq N} q^{n-l_n}$ and $c_0 > 0$ is some constant.
Thanks for Your Attention!