# Refined Asymptotics for the Number of Leaves of Random Point Quadtrees

Michael Fuchs
Department of Applied Mathematics
National Chiao Tung University
Hsinchu 300
Taiwan

Noela S. Müller
Institute for Mathematics
Goethe University
60054 Frankfurt a.M.
Germany

Henning Sulzbach
University of Birmingham
School of Mathematics
B15 2TT, Birmingham
United Kingdom

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**Abstract**

In the early 2000s, several phase change results from distributional convergence to distributional non-convergence have been obtained for shape parameters of random discrete structures. Recently, for those random structures which admit a natural martingale process, these results have been considerably improved by obtaining refined asymptotics for the limit behavior. In this work, we propose a new approach which is also applicable to random discrete structures which do not admit a natural martingale process. As an example, we obtain refined asymptotics for the number of leaves in random point quadtrees. More applications, for example to shape parameters in generalized $m$-ary search trees and random gridtrees, will be discussed in the journal version of this extended abstract.

## 1 Introduction and Result

In this extended abstract, we investigate shape parameters of random discrete structures whose distributional behavior is known to undergo a phase change as a structural characteristic of the structure varies. Several such phase change phenomena, in particular with a change from distributional convergence to distributional non-convergence, have been found in the early 2000s. We start by recalling a particular nice and surprising result in this direction which was obtained by Janson in [11]: the phase change of the number of nodes with depth in a fixed congruent class in random recursive trees.

First, we recall the definition of random recursive trees. Starting from a root, nodes are added consecutively where the $n$-th node is attached uniformly at random as left-most child to one of the existing nodes. In such a tree with $n$ nodes, let $M_n$ denote the number of nodes with depth (distance from the root) divisible by $m$ where $m \geq 2$ is fixed. Set

$$\omega_n = \begin{cases} n, & \text{if } 6 \nmid m; \\ n \log n, & \text{if } 6 \mid m. \end{cases}$$

Then, in [11], the following result was proved: if $2 \leq m \leq 6$, we have

$$\frac{M_n - n/m}{\sqrt{\omega_n}} \xrightarrow{d} N(0, \sigma_m^2), \tag{1}$$

where $\sigma_m > 0$; for all other $m$, we have that $M_n$ with the standard normalization, i.e., $(M_n - n/m)/\sqrt{\mathrm{Var}(M_n)}$, does not converge to a fixed limit law.

A similar result holds if the depths of nodes are required to fall into another residue class. Moreover, the same phase change phenomenon is present in random binary search trees, too; see [11]. Also, several other shape parameters in diverse families of random trees have been proved to exhibit a similar phase change behavior from distributional convergence to distributional non-convergence, e.g., the size of $m$-ary search trees proved by Chern and Hwang [3] (see also Mahmoud and Pittel [14] and Lew and Mahmoud [13] for preliminary results) and the number of leaves in random point quadtrees proved by Chern et al. [1]; see Table 1 for a summary of these results and [1, 3] for many more examples.

| structure | parameter | non-convergence | refined asymptotics |
|---|---|---|---|
| recursive trees | nodes with depths divisible by $m$ | $m \geq 7$ | [16, 17] |
| $m$-ary search trees | size | $m \geq 27$ | [15] |
| $d$-dimensional quadtrees | number of leaves | $d \geq 9$ | this paper |

Table 1: *A summary of shape parameters and discrete structures for which the distributional behavior changes from normal to non-convergence.*

After the above results have been published, subsequent research has focused on clarifying the stochastic behavior in the non-convergence regime; e.g. see [1], Chern et al. [2], Chauvin and Pouyanne [4], Fill and Kapur [6], and [11]. This line of research has recently culminated in the realization that subtracting a sufficiently large number of suitable random variables leads to a central limit theorem. To give some more details, consider again the above random variable $M_n$. Set $r = \lfloor (m-1)/6 \rfloor$ and

$$\zeta_k := \cos\left(\frac{2\pi k}{m}\right) \qquad \text{and} \qquad \eta_k := \sin\left(\frac{2\pi k}{m}\right).$$

Following a technique developed by Neininger [18] in a refined analysis of the complexity of the randomised Quicksort algorithms, it was proved by the second author of this extended abstract and Neininger [16, 17] that there exist complex random variables $\Xi_1, \ldots, \Xi_r$ such that

$$\frac{1}{\sqrt{\omega_n}}\left(M_n - \frac{n}{m} - 2\sum_{k=1}^{r} \Re\left(\Xi_k n^{i\eta_k}\right) n^{\zeta_k}\right) \xrightarrow{d} N(0, \sigma_m^2)$$

with $\sigma_m > 0$. Note that this result yields (1) as a special case.

The proof of the above result made use of a natural martingale process related to random recursive trees. Moreover, another proof method (also using the martingale process) was proposed by the second author in [15], where the above result was extended to generalized Pólya urns; see Janson [10] for background. The latter result contains both the above result and a similar result for $m$-ary search trees; see Table 1.

The purpose of this work is to propose yet another approach which does not make use of the martingale process (the possibility of such an approach was already announced in [17]). The advantage of such a method is that it can be applied to random discrete structures which do not admit such a process. This is for instance the case for random point quadtrees which we use in this work as guiding example. Other applications of our approach in the context of, e.g., generalized $m$-ary search trees and gridtrees (where there are again no natural martingale processes) will be discussed in the journal version of this extended abstract.

We first recall the definition of random point quadtrees (which for brevity will be called random quadtrees in the sequel). Fix a dimension $d$ and consider an infinite sequence of stochastically independent points chosen uniformly at random from the $d$-dimensional unit cube. Then, the first point is stored in the root which has $2^d$ subtrees that correspond to the $2^d$ quadrants into which the $d$-dimensional unit cube is split by the first point. These subtrees contain the points which fall into these quadrants respectively. Moreover, subtrees are built recursively via the same process. The resulting tree after $n$ steps is called random quadtree of size $n$.

In such a tree of size $n$, let $L_n$ denote the number of leaves. Then, in [1], the following phase change result was proved: if $1 \le d \le 8$, then

$$\frac{L_n - \kappa_d n}{\sqrt{n}} \xrightarrow{d} N(0, \sigma_d^2),$$

where $\sigma_d > 0$ and

$$\kappa_d = 1 - \frac{2}{d}\xi'(1), \tag{2}$$

where $\xi(s)$ is given in (4); for all other $d$, we have that $L_n$ with the standard normalization does not converge to a fixed limit law. (For $d = 1$, the result goes back to Devroye [5].)

The main result of this extended abstract is the following extension of this result which gives an asymptotic expansion of the limit behavior in the style of [15, 16, 17].

**Theorem 1.1.** *Let $d \ge 1$. Then, there exist complex random variables $\mathcal{Z}_1, \ldots, \mathcal{Z}_p$ such that*

$$\frac{1}{\sqrt{n}}\left(L_n - \kappa_d n - 2\sum_{k=1}^{p} \Re(\mathcal{Z}_k n^{i\beta_k}) n^{\alpha_k}\right) \xrightarrow{d} N(0, \sigma_d^2), \tag{3}$$

*where $\sigma_d > 0$. Here,*

$$\alpha_k := 2\cos\left(\frac{2\pi k}{d}\right) - 1 \qquad \textit{and} \qquad \beta_k := 2\sin\left(\frac{2\pi k}{d}\right)$$

*and $p$ is the largest number in $\{0, \ldots, \lfloor d/2 \rfloor\}$ with $\alpha_k > 1/2$; see Table 2.*

We conclude the introduction with a discussion of the proof of Theorem 1.1 and an outline of the manuscript. Following [18, 16], the proof relies on the following three steps:

| $d$ | $1, \ldots, 8$ | $9, \ldots, 17$ | $18, \ldots, 26$ | $27, \ldots, 34$ |
|---|---|---|---|---|
| $p$ | 0 | 1 | 2 | 3 |

Table 2: *Value of $p$ in (3) for small values of $d$.*

(i) the construction of the limiting random variables $\mathcal{Z}_1, \ldots, \mathcal{Z}_p$,

(ii) an expansion of the variance of the residual $L_n - \kappa_d n - 2\sum_{k=1}^{p} \Re(\mathcal{Z}_k n^{i\beta_k}) n^{\alpha_k}$, and

(iii) general techniques to deduce the asymptotic normality (3) from (ii) from a distributional recurrence for the sequence of residuals.

In the literature, step (iii) in the present context has been carried out relying on two different techniques which both apply with straightforward modifications in our setting: the contraction method [18, 16] and the method of moments [9]. As this part does not involve significantly new arguments, we refrain from discussing the details in this extended abstract and refer the reader to the journal version of this work (to come).

The remainder of the manuscript is organized as follows. In Section 2, we give an explicit construction of the quadtree sequence and state known asymptotic expansions for the mean number of leaves. Section 3 is dedicated to step (i) and uses contraction arguments; the proofs are found in Appendix A.

The most technical part of the work, step (ii), crucially relies on a recursive distributional decomposition of the residual sequence and asymptotic transfer theorems developed by Chern, Fuchs and Hwang [1] for general parameters in quadtrees. This part, worked out in Section 4, is based on conceptually novel ideas since second moments cannot be computed by direct means exploiting a martingale structure. Proofs of technical lemmas required here are collected in the Appendix B.


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# 2 Preliminaries

Let us start with an explicit construction of the quadtrees. To this end, let $Y^{(i)}, i \geq 1$ be a sequence of independent random variables following the uniform distribution on $[0,1]^d$. We define a sequence of trees $T_0, T_1, \ldots$ where $T_i$ stores the values $Y^{(1)}, \ldots, Y^{(i)}$ as follows: initially, we start with an empty tree $T_0$ consisting of a placeholder associated with the unit cube. $Y^{(1)}$ replaces the placeholder thereby creating a tree $T_1$ consisting of a root node to which we associate $2^d$ placeholders which are assigned the $2^d$ rectangular regions in which the components of $Y^{(1)}$ partition the unit cube. (In computer science, these placeholders are often called external nodes.) Inductively, having constructed the tree $T_n$ storing $Y^{(1)}, \ldots, Y^{(n)}$ with $1 + (2^d - 1)n$ placeholders corresponding to $1 + (2^d - 1)n$ regions partitioning the unit cube, we obtain the tree $T_{n+1}$ by storing $Y^{(n+1)}$ in the placeholder associated with the rectangle containing $Y^{(n+1)}$.

We let $L_n$ denote the number of leaves in the random quadtree $T_n$. Set $\mu_n := \mathbb{E}[L_n]$. To describe the asymptotic behavior of $\mu_n$, it is necessary to introduce some terminology from [7]:

first, for $s \in \mathbb{C} \setminus \{0\}$, let $[s] := 1 - \frac{2^d}{s^d}$. Then, for $n \in \mathbb{N}, n \geq 3$, we define the $d$-analogue of the factorial as

$$[n]! := [3] \cdot [4] \cdots [n] \qquad \text{and} \qquad [2]! := 1.$$

Let $A := \{2\omega^k - j : k \in \{0, \ldots, d-1\}, j \in \mathbb{N}\}$. The definition of $[n]!$ extends holomorphically to complex numbers $s \in \mathbb{C} \setminus A$ through

$$[s]! := \prod_{j=1}^{\infty} \frac{[j+2]}{[j+s]}, \qquad \text{and } [\infty]! := [3] \cdot [4] \cdot [5] \cdots .$$

Flajolet et al. [7], showed that, for all $n \geq 2$,

$$\mu_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \mu_k^* \quad \text{with} \quad \mu_k^* = \begin{cases} 0, & k = 0 \\ -1, & k = 1 \\ -\sum_{j=2}^{k} \frac{[k]!}{[j]!}, & k \geq 2. \end{cases}$$

From here, standard techniques relying on Nörlund-Rice integrals for meromorphic functions arising in the analysis of finite differences such as, e.g. [8, Theorem 2], allow to derive asymptotic expansions of $\mu_n$ (as $n \to \infty$) of arbitrary precision. In particular, following the notation in [7], with

$$\xi(s) := \frac{s-1}{[\infty]!} + \sum_{j=2}^{\infty} \left( \frac{1}{[j]!} - \frac{1}{[s+j-1]!} \right), \tag{4}$$

one finds

$$\mu_n = \kappa_d n - 2 \sum_{1 \leq k \leq \lfloor d/2 \rfloor, \alpha_k > 0} \Re\left(\gamma_k n^{i\beta_k}\right) n^{\alpha_k} + \mathcal{O}(1), \tag{5}$$

where $\kappa_d$ is given in (2) and, with $\lambda_k = \alpha_k + i\beta_k$ for $k = 1, \ldots, \lfloor d/2 \rfloor$ with $\alpha_k > 0$,

$$\gamma_k = -\frac{\lambda_k + 1}{d} \Gamma(-\lambda_k) \xi(\lambda_k) [\lambda_k + 1]!.$$

Here, $\Gamma(\cdot)$ denotes the Gamma function. The details of the argument show that $\gamma_k \neq 0$ for all $k \geq 1, \alpha_k > 0$, so no term in the asymptotic expansion (5) vanishes. For later purposes, note that $\alpha_k \neq 1/2$ for all $k = 1, \ldots, d-1$, since the converse would imply the existence of a $k$-th root of unity with real part $3/4$. But any rational real part of a root of unity takes values in the set $\{0, -1/2, 1/2, 1, -1\}$ since, with $\omega := \alpha_1 + i\beta_1$, the value $2\Re(\omega^k) = \omega^k + \omega^{d-k}$ is an algebraic integer for any $1 \leq k \leq d$.

# 3 A family of limiting random variables

As opposed to the applications discussed in [18, 16, 17], there is no natural martingale process associated with the sequence $L_n, n \geq 1$. Therefore, it is necessary to construct the limiting random variables $\mathcal{Z}_1, \ldots, \mathcal{Z}_p$ in Theorem 1.1 in an ad-hoc way guided by the recursive distributional decomposition of $L_n$. In this section, we give the details of this construction.

Let $\mathbb{T}$ be the complete infinite $2^d$-ary tree represented in standard Ulam-Harris notation by

$$\mathbb{T} = \bigcup_{i \geq 0} \{0, \dots, 2^d - 1\}^i.$$

Through the canonical embedding of the sequence $T_0, T_1, \dots$ of increasing trees into $\mathbb{T}$, to any $v \in \mathbb{T}$, we shall associate a random integer $\ell(v) \geq 1$ such that $Y^{(\ell(v))}$ is stored at node $v$. (Clearly, as the fill-up level of $T_n$ grows to infinity, $\ell(v)$ exists for all nodes $v \in \mathbb{T}$.) For $\ell \geq 1$, let $I_\ell$ be the rectangle corresponding to the placeholder in $T_{\ell-1}$ which contains $Y^{(\ell)}$. Define $\tilde{\mathcal{Y}}^{(\ell)}$ as the vector of components of $Y^{(\ell)}$ relative to the boundaries of $I_\ell$. Formally, with $I_\ell = [i_1^\wedge, i_1^\vee] \times \cdots \times [i_d^\wedge, i_d^\vee]$, we set

$$\tilde{\mathcal{Y}}_k^{(\ell)} = \frac{Y_k^{(\ell)} - i_k^\wedge}{i_k^\vee - i_k^\wedge}, \quad k = 1, \dots, d.$$

Finally, for $v \in \mathbb{T}$, let

$$U^{(v)} := \tilde{\mathcal{Y}}^{(\ell(v))}.$$

By construction, $\{U^{(v)} : v \in \mathbb{T}\}$ is a family of independent random variables with the uniform distribution on $[0,1]^d$. While the placeholders associated with the nodes in the tree $T_n$ give rise to a partition of the unit cube, the construction of the limiting random variables relies on different decompositions of the unit cube traversing $\mathbb{T}$ level-wise. To this end, to every $v \in \mathbb{T}$ and $0 \leq j \leq 2^d - 1$, writing $j = \sum_{\ell=1}^d \varepsilon_\ell 2^{\ell-1}$ with $\varepsilon_1, \dots, \varepsilon_d \in \{0,1\}$, we associate the random variables $\Delta_j^{(v)} := V_1^{(v)} \cdots V_d^{(v)}$, where

$$V_\ell^{(v)} := \begin{cases} U_\ell^{(v)}, & \text{if} \quad \varepsilon_\ell = 0 \\ 1 - U_\ell^{(v)}, & \text{if} \quad \varepsilon_\ell = 1. \end{cases}$$

Note that $\sum_{j=0}^{2^d-1} \Delta_j^{(v)} = 1$. Subsequently, write $\Delta^{(v)} = (\Delta_0^{(v)}, \dots, \Delta_{2^d-1}^{(v)})$.

Let $k \in \{1, \dots, d-1\}$ with $\alpha_k > 1/2$ and define a family of random variables $\{\mathcal{Z}_{n,k}^{(v)} : n \geq 0, v \in \mathbb{T}\}$ as follows: first, set $\mathcal{Z}_{0,k}^{(v)} = \gamma_k$ for all $v \in \mathbb{T}$. Then, for $n \geq 1$ and $v \in \mathbb{T}$, we recursively define

$$\mathcal{Z}_{n,k}^{(v)} := \sum_{j=0}^{2^d-1} \left( \Delta_j^{(v)} \right)^{\lambda_k} \cdot \mathcal{Z}_{n-1,k}^{(vj)}.$$

Note that, for all $n \geq 0$, we have $\mathcal{Z}_{n,d-k}^{(v)} = \overline{\mathcal{Z}_{n,k}^{(v)}}$. Let $\Pi_\emptyset := 1$, and, recursively, for $v \in \mathbb{T}$ and $j = 0, \dots, 2^d - 1$,

$$\Pi_{vj} = \Delta_j^{(v)} \Pi_v.$$

Then, we have the following forward expression for $\mathcal{Z}_{n,k}^{(\emptyset)}$ :

$$\mathcal{Z}_{n,k}^{(\emptyset)} = \gamma_k \sum_{|v|=n} \Pi_v^{\lambda_k}.$$

Analogous expansions can be stated for $\mathcal{Z}_{n,k}^{(v)}, v \in \mathbb{T}$. Let $\mathcal{F}_{-1}$ be the trivial $\sigma$-field, and, for $n \geq 0$, set $\mathcal{F}_n = \sigma\left(U^{(v)} : v \in \mathbb{T}, |v| \leq n\right)$. It follows immediately from the previous display that $\mathcal{Z}_{n,k}^{(\emptyset)}, n \geq 0$ is a martingale with respect to the filtration $\mathcal{F}_n, n \geq -1$.

This martingale has the following important property.

**Proposition 3.1.** *For all $v \in \mathbb{T}$ there exists a random variable $\mathcal{Z}_k^{(v)}$ such that, almost surely and with respect to all moments,*

$$\mathcal{Z}_{n,k}^{(v)} \to \mathcal{Z}_k^{(v)}. \tag{6}$$

*We have*

*(i) the random variables $\mathcal{Z}_k^{(v)}, v \in \mathbb{T}$ are identically distributed,*

*(ii) $\mathcal{Z}_k^{(v0)}, \ldots, \mathcal{Z}_k^{(v(2^d-1))}, \Delta^{(v)}$ are stochastically independent and*

$$\mathcal{Z}_k^{(v)} = \sum_{j=0}^{2^d-1} \left(\Delta_j^{(v)}\right)^{\lambda_k} \cdot \mathcal{Z}_k^{(vj)},$$

*(iii) the law of $\mathcal{Z}_k^{(\emptyset)}$ is the unique distribution satisfying $\mathbb{E}[\mathcal{Z}_k^{(\emptyset)}] = \gamma_k, \mathbb{E}[|\mathcal{Z}_k^{(\emptyset)}|^2] < \infty$ and*

$$\mathcal{Z}_k^{(\emptyset)} \stackrel{d}{=} \sum_{j=0}^{2^d-1} \left(\Delta_j^{(\emptyset)}\right)^{\lambda_k} \cdot \mathcal{Z}_k^{*,(j)}, \tag{7}$$

*where $\mathcal{Z}_k^{*,(0)}, \ldots, \mathcal{Z}_k^{*,(2^d-1)}$ are independent copies of $\mathcal{Z}_k^{*,(\emptyset)}$, independent of $\Delta^{(\emptyset)}$.*

In the remainder of the manuscript, we agree to drop the upper index $\emptyset$ when referring to the quantities $\mathcal{Z}_k^{(\emptyset)}, k = 1, \ldots, p$ and $\Delta_j^{(\emptyset)}, j = 0, \ldots, 2^d - 1$ and $U_j^{(\emptyset)}, j = 1, \ldots, d$.

Below, we will need the following property of the $\mathcal{Z}_k$'s which follows from Leckey [12].

**Proposition 3.2.** *Let $1 \leq k \leq p$. The vector $(\Re(\mathcal{Z}_k), \Im(\mathcal{Z}_k))$ has a Schwartz density $f$ on $\mathbb{R}^2$, that is, $f$ is infinitely differentiable, where $f$ and all its derivatives decay faster to zero at infinity than any polynomial.*

# 4 The variance of the residual

In the final chapter of the manuscript, we discuss the techniques to prove step (ii) outlined in the introduction. Let

$$L_n^* := L_n - \kappa_d n - 2 \sum_{k=1}^{p} \Re\left(\mathcal{Z}_k n^{i\beta_k}\right) n^{\alpha_k} + \delta_n,$$

where $\delta_n$ is deterministic such that $\mathbb{E}[L_n^*] = 0$. (Exact scaling simplifies arguments in the following.) By (5), we have $\delta_n = \mathcal{O}\left(n^{\max(\alpha_{p+1},0)}\right)$. (One actually has $\alpha_{p+1} > 0$ for all $d > 11$.) For $j = 0, \ldots, 2^d - 1$, let $N_j$ be the size of the $j$-th subtree of the root and $L_n^{(j)}$ be the number of leaves it contains. Given $\Delta_0, \ldots, \Delta_{2^d-1}$, the vector $(N_0, \ldots, N_{2^d-1})$ has the multinomial distribution with parameter $(n-1; \Delta_0, \ldots, \Delta_{2^d-1})$. We now set up a distributional recurrence for $L_n^*$. As $\mathcal{Z}_k = \sum_{j=0}^{2^d-1} \Delta_j^{\lambda_k} \mathcal{Z}_k^{(j)}$ it follows that

$$\begin{aligned} L_n^* &= \sum_{j=0}^{2^d-1} \left( L_n^{(j)} - \kappa_d N_j + \delta_{N_j} - 2 \sum_{k=1}^{p} \Re\left(\mathcal{Z}_k^{(j)} N_j^{i\beta_k}\right) N_j^{\alpha_k} \right) + r_n + D_n \\ &=: \sum_{j=0}^{2^d-1} \mathbb{L}_n^{(j)} + r_n + D_n, \end{aligned} \tag{8}$$

where

$$r_n := \delta_n - \sum_{j=0}^{2^d-1} \delta_{N_j} - \kappa_d, \quad \text{and } D_n := 2 \sum_{j=0}^{2^d-1} \sum_{k=1}^{p} \Re\left(\mathcal{Z}_k^{(j)}\left(N_j^{\lambda_k} - (\Delta_j n)^{\lambda_k}\right)\right).$$

By the construction of the quadtree, $\left(\mathbb{L}_n^{(0)}, \dots, \mathbb{L}_n^{(2^d-1)}\right) \stackrel{d}{=} \left(\bar{L}_{N_0}^{(0)*}, \dots, \bar{L}_{N_{2^d-1}}^{(2^d-1)*}\right)$, where $\left(\bar{L}_k^{(0)*}\right)_{k\geq 0}, \dots, \left(\bar{L}_k^{(2^d-1)*}\right)_{k\geq 0}$ are independent copies of the process $(L_k^*)_{k\geq 0}$. Note that $(N_0, \dots, N_{2^d-1})$ and $\{U^{(v)} : v \in \mathbb{T} \setminus \{\emptyset\}\}$ are independent. Note however, that $D_n$ and $\left(\mathbb{L}_n^{(0)}, \dots, \mathbb{L}_n^{(2^d-1)}\right)$ are not stochastically independent, not even given $(\Delta, N_0, \dots, N_{2^d-1})$, since both quantities involve $\mathcal{Z}_k^{(j)}, j = 0, \dots, 2^d - 1, k = 1, \dots, p$.

## 4.1 An asymptotic expansion for the variance of $L_n^*$

The remainder of this extended abstract is devoted to the proof of the following proposition.

**Proposition 4.1.** *There exists $0 < \sigma_d < \infty$ such that, as $n \to \infty$,*

$$\mathrm{Var}(L_n^*) = \sigma_d n + o(n).$$

Of course, as $\delta_n = o(\sqrt{n})$, the same asymptotic expansion applies to the variance of the residual sequence $L_n^* - \delta_n$. To prove the proposition, note that, from (8), straightforward calculations reveal that, with $a(n) := \mathbb{E}\left[(L_n^*)^2\right]$, we have

$$\begin{aligned} a(n) &= 2^d \mathbb{E}[a(N_0)] + \mathbb{E}\left[r_n^2\right] + \mathbb{E}\left[D_n^2\right] + 2\mathbb{E}\left[D_n r_n\right] + 2\mathbb{E}\Big[D_n \sum_{j=0}^{2^d-1} \mathbb{L}_n^{(j)}\Big] \\ &=: 2^d \mathbb{E}[a(N_0)] + b(n). \end{aligned} \tag{9}$$

This is the quadtree recurrence (see the lemma below). Our aim is to apply the asymptotic transfer theorems for it developed by Chern, Fuchs and Hwang [1]. To this end, we need to understand the asymptotic behavior of the additive sequence $b(n)$ in the last display. In particular, we would like to use the following result from [1].

**Theorem 4.2** ([1], Theorem 2(i))**.** *Consider the quadtree recurrence*

$$a_n = b_n + 2^d \sum_{0\leq j<n} \pi_{n,j} a_j, \qquad (n \geq 1),$$

*where $a_0 = 0$ and*

$$\pi_{n,j} = \mathbb{P}\left(N_0 = j\right) = \binom{n-1}{j} \int_0^1 u^j (1-u)^{n-1-j} \frac{(-\log u)^{d-1}}{(d-1)!} \mathrm{d}u.$$

*If $b_n = o(n)$ and the series $\sum_{n\geq 1} b_n/n^2$ converges, then $a_n = \kappa n + o(n)$ for some $\kappa \in \mathbb{R}$.*

For infinite sum representations of the limiting constant $\kappa$, we refer to [1]. The theorem does not exclude the case that $\kappa = 0$, which explains the necessity of the following lemma, whose proof is deferred to the Appendix B.

**Lemma 4.3.** *In the set-up of the previous theorem, assume that*

*(a) (i) $b_n$ is non-negative for all $n$, and (ii) $b_n$ is positive for at least one $n$, or*

*(b) (i) $a_n$ is non-negative for all $n$, and (ii) $b_n$ is positive for all $n$ large enough.*

*Then, $a_n = \Omega(n)$.*

We also need the following two lemmas, where the first is a straightforward implication of the multivariate central limit theorem for $(N_0, \ldots, N_{2^d-1})$, while the technical proof of the second lemma is given in the Appendix B.

**Lemma 4.4** (Multivariate central limit theorem)**.** *Let $z \in \mathbb{C}$ with $1/2 \leq \Re(z) < 1$. In distribution, in $\mathbb{C}^{2^d}$,*

$$\left(\frac{N_0^z - (\Delta_0 n)^z}{n^{z-1/2}}, \ldots, \frac{N_{2^d-1}^z - (\Delta_{2^d-1} n)^z}{n^{z-1/2}}\right) \to X,$$

*where $X_i = z \cdot \Delta_i^{z-1} Y_i$ with $Y = \Sigma^{1/2} \cdot \mathcal{N}$, where $\mathcal{N} = (\mathcal{N}_0, \ldots, \mathcal{N}_{2^d-1})$ has the standard multivariate normal distribution, $(\Delta_0, \ldots, \Delta_{2^d-1})$ and $\mathcal{N}$ are stochastically independent, and the covariance matrix $\Sigma$ satisfies*

$$\Sigma_{i,j} = \begin{cases} \Delta_i(1-\Delta_i) & \text{if } i = j, \\ -\Delta_i \Delta_j & \text{if } i \neq j. \end{cases}$$

**Lemma 4.5.** *We have the following asymptotic expansions:*

*(i) for any $z \in \mathbb{C}$ with $0 < \Re(z) < 1$ and $\varepsilon > 0$, we have, as $n \to \infty$,*

$$\mathbb{E}[N_0^z] = \mathbb{E}\left[\Delta_0^z\right] n^z + \frac{z(z-1)}{2} \mathbb{E}\left[(1-\Delta_0)\, \Delta_0^{z-1}\right] n^{z-1} + \mathcal{O}(n^{\varepsilon - 1}).$$

*(ii) For any $z \in \mathbb{C}$ with $1/2 < \Re(z) < 1$ and fixed $p \in \mathbb{N} \setminus \{0\}$, we have*

$$\|N_0^z - (\Delta_0 n)^z\|_p = |z| \left\| \Delta_0^{\Re(z)/2} \sqrt{1-\Delta_0} \right\|_p \|\mathcal{N}_0\|_p \, n^{\Re(z)-1/2} + o(n^{\Re(z)-1/2}).$$

*(iii) For any $z \in \mathbb{C}$ with $0 < \Re(z) < 1/2$ and fixed $p \in \mathbb{N} \setminus \{0\}$, we have*

$$\|N_0^z - (\Delta_0 n)^z\|_p = \mathcal{O}(1).$$

The first step to show Proposition 4.1 is to verify that the contribution of the mixed term in $b(n)$ is asymptotically negligible.

**Lemma 4.6.** *As $n \to \infty$, we have $\mathbb{E}\left[D_n \sum_{j=0}^{2^d-1} \mathbb{L}_n^{(j)}\right] = \mathcal{O}(n^{\alpha_1 - 1/2})$.*

*Proof.* First of all, note that $\mathbb{E}\left[r_n^2\right] = \mathcal{O}(n^{2\max(\alpha_{p+1},0)})$ since $\delta_n = \mathcal{O}\left(n^{\max(\alpha_{p+1},0)}\right)$ and $N_j \leq n$ for all $j = 0, \ldots, 2^d - 1$. As $\mathcal{Z}_k^{(j)}$ and $(N_j, \Delta_j)$ are stochastically independent, it follows from part *(ii)* of the previous lemma that

$$\mathbb{E}\left[\left|\mathcal{Z}_k^{(j)}\left(N_j^{\lambda_k} - (\Delta_j n)^{\lambda_k}\right)\right|^2\right] = \mathbb{E}\left[|\mathcal{Z}_k|^2\right] \mathbb{E}\left[\left|N_0^{\lambda_k} - (\Delta_0 n)^{\lambda_k}\right|^2\right] = \mathcal{O}(n^{2\alpha_k - 1}).$$

A standard application of the Cauchy-Schwarz inequality shows that $\mathbb{E}\left[D_n^2\right] = \mathcal{O}(n^{2\alpha_1-1})$. Next, by independence of quantities defined in subtrees, we obtain

$$\mathbb{E}\Big[D_n \sum_{j=0}^{2^d-1} \mathbb{L}_n^{(j)}\Big] = 2\sum_{j=0}^{2^d-1} \mathbb{E}\Big[\mathbb{L}_n^{(j)} \sum_{k=1}^{p} \Re\left(\mathcal{Z}_k^{(j)}\left(N_j^{\lambda_k} - (\Delta_j n)^{\lambda_k}\right)\right)\Big]. \tag{10}$$

Conditionally on $\{N_0 = n_0, \dots, N_{2^d-1} = n_{2^d-1}\}$ where $n_0 + \dots + n_{2^d-1} = n-1$, we have

(i) the random variables $(\Delta_0, \dots, \Delta_{2^d-1}), (\mathcal{Z}_1^{(j)}, \dots, \mathcal{Z}_p^{(j)}, \mathbb{L}_n^{(j)})$ are stochastically independent, and

(ii) $(\mathcal{Z}_1^{(j)}, \dots, \mathcal{Z}_p^{(j)}, \mathbb{L}_n^{(j)})$ is distributed like $(\mathcal{Z}_1, \dots, \mathcal{Z}_p, L_{n_j}^*)$.

To estimate (10), consider the terms

$$\begin{aligned}\mathbb{E}\left[\mathbb{L}_n^{(j)}\Re(\mathcal{Z}_k^{(j)}(N_j^{\lambda_k} - (\Delta_j n)^{\lambda_k}))\right] &= \sum_{\ell=0}^{n-1} \mathbb{P}\left(N_j = \ell\right)\mathbb{E}\left[L_\ell^*\Re(\mathcal{Z}_k)\right]\mathbb{E}\left[\Re\left(\ell^{\lambda_k} - (\Delta_j n)^{\lambda_k}\right)\right] \\ &\quad - \sum_{\ell=0}^{n-1} \mathbb{P}\left(N_j = \ell\right)\mathbb{E}\left[L_\ell^*\Im(\mathcal{Z}_k)\right]\mathbb{E}\left[\Im\left(\ell^{\lambda_k} - (\Delta_j n)^{\lambda_k}\right)\right].\end{aligned}$$

By the trivial bound $\mathbb{E}[(L_n^*)^2] = \mathcal{O}(n^2)$, it follows from the Cauchy-Schwarz inequality that there exists a constant $C > 0$ such that

$$\max\left\{\mathbb{E}\left[L_n^*\Re(\mathcal{Z}_k)\right], \mathbb{E}\left[L_n^*\Im(\mathcal{Z}_k)\right]\right\} \leq Cn.$$

Therefore,

$$\left|\mathbb{E}\left[\mathbb{L}_n^{(j)}\Re(\mathcal{Z}_k^{(j)}(N_j^{\lambda_k} - (\Delta_j n)^{\lambda_k}))\right]\right| \leq 2Cn\left|\mathbb{E}\left[N_j^{\lambda_k} - (\Delta_j n)^{\lambda_k}\right]\right|. \tag{11}$$

From part *(i)* of the previous lemma, it follows that the right hand side of (11) grows at most of the order $n^{\alpha_k}$. Overall, this shows that

$$\mathbb{E}\Big[D_n \sum_{j=0}^{2^d-1} \mathbb{L}_n^{(j)}\Big] = \mathcal{O}(n^{\alpha_1}).$$

Combining the bounds on $\mathbb{E}\left[r_n^2\right], \mathbb{E}\left[D_n^2\right]$ and the last display, Theorem 4.2 yields $\mathrm{Var}(L_n^*) = \mathcal{O}(n)$. Repeating the last steps using this improved bound concludes the proof. ∎

The previous proposition suggests that the order of magnitude of the additive term in (9) is $\max\{n^{2\max(\alpha_{p+1},0)}, n^{2\alpha_1-1}\}$. For most values of $d$, we have $2\alpha_1 - 1 > 2\alpha_{p+1}$. Indeed, for $9 \leq d \leq 10,000$, there exist only 31 values ranging from $d = 15$ to $d = 8598$ for which the converse is true. It is important to note that, for all $d \geq 9$, we have $2\alpha_1 - 1 \neq 2\alpha_{p+1}$ since the contrary would imply that $\omega + \omega^{d-1} - \omega^{p+1} - \omega^{d-p-1} = 1/2$ which is impossible since the left hand side is an algebraic integer. In particular, in light of Theorem 4.2 and Lemma 4.3, the following two propositions verifying that $b(n) \to \infty$ are the missing pieces to conclude the proof of Proposition 4.1.

**Proposition 4.7.** *Let $\alpha_{p+1} > 0$, that is, $d > 11$ and*

$$W := \sum_{i=0}^{2^d-1} \Delta_i^{\lambda_{p+1}} = \prod_{i=1}^{d} \left( U_i^{\lambda_{p+1}} + (1-U_i)^{\lambda_{p+1}} \right).$$

*For $x \in \mathbb{R}$, let*

$$\Phi(x) := 2\Re \left( \gamma_{p+1}^2 \mathbb{E}\left[(1-W)^2\right] e^{2i\beta_{p+1}x} \right) + 2|\gamma_{p+1}|^2 \mathbb{E}\left[|1-W|^2\right].$$

*$\Phi$ is a smooth periodic function with period $\pi/\beta_{p+1}$, amplitude $2|\gamma_{p+1}|^2|\mathbb{E}\left[(1-W)^2\right]|$ and*

$$\min_{x\in\mathbb{R}} \Phi(x) = 2|\gamma_{p+1}|^2 \left[ \mathbb{E}\left[|1-W|^2\right] - \left|\mathbb{E}\left[(1-W)^2\right]\right| \right] > 0.$$

*As $n \to \infty$,*

$$\mathbb{E}\left[r_n^2\right] = \Phi(\log n) n^{2\alpha_{p+1}} + \mathcal{O}(n^{\alpha_{p+1}+\alpha_{p+2}}).$$

**Proposition 4.8.** *Let $(\Delta, Y)$ be as in Lemma 4.4 and stochastically independent of $\mathcal{Z}_1^{(0)}, \ldots, \mathcal{Z}_1^{(2^d-1)}$. Set*

$$\mathcal{W} = \sum_{j=0}^{2^d-1} \lambda_1 \mathcal{Z}_1^{(j)} \Delta_j^{\lambda_1 - 1} Y_j.$$

*For $x \in \mathbb{R}$, define*

$$\Psi(x) := 2\Re \left( \mathbb{E}\left[\mathcal{W}^2\right] e^{2i\beta_{p+1}x} \right) + 2\mathbb{E}\left[|\mathcal{W}|^2\right].$$

*$\Psi$ is a smooth periodic function with period $\pi/\beta_{p+1}$, amplitude $2|\mathbb{E}\left[\mathcal{W}^2\right]|$ and*

$$\min_{x\in\mathbb{R}} \Psi(x) = 2\left( \mathbb{E}\left[|\mathcal{W}|^2\right] - \left|\mathbb{E}\left[\mathcal{W}^2\right]\right| \right) > 0.$$

*As $n \to \infty$, we have*

$$\mathbb{E}[D_n^2] = \Psi(\log n) n^{2\alpha_1 - 1} + o(n^{2\alpha_1-1}).$$

The proofs of these propositions are very similar and we only present the proof of Proposition 4.8 which is more involved.

*Proof of Proposition 4.8.* By definition, $\Psi$ has period $\pi/\beta_{p+1}$. Next, for any $z \in \mathbb{C}$, it is easy to see that the global maximum and minimum of the function $x \mapsto \Re(z \exp(ix))$ are $|z|$ and $-|z|$. This implies the remaining claims on the shape of $\Psi$. $\min_{x\in\mathbb{R}} \Psi(x) > 0$ follows from triangle inequality upon verifying that $\arg(\mathcal{W})$ is not almost surely constant. This, in turn follows from that fact that, for any given (affine) line $L \subseteq \mathbb{C}$, we have $\mathbb{P}\left(\mathcal{Z}_1 \in L\right) = 0$. This is an immediate corollary of the fact that $(\Re(\mathcal{Z}_1), \Im(\mathcal{Z}_1))$ admits a density on $\mathbb{R}^2$ (see Proposition 3.2). For the asymptotic expansion of $D_n$, note that, following the steps involving the Cauchy-Schwarz inequality and the bounds stated in the proof of Proposition 4.6, it is straightforward to verify that

$$\mathbb{E}\left[D_n^2\right] = 4\mathbb{E}\left[ \left( \sum_{j=0}^{2^d-1} \Re\left( \mathcal{Z}_1^{(j)} \left( N_j^{\lambda_1} - (n\Delta_j)^{\lambda_1} \right) \right) \right)^2 \right] + \mathcal{O}(n^{\alpha_1+\alpha_2-1}).$$

By the multivariate central limit theorem stated in Lemma 4.4, the first term is asymptotically equivalent to $\Psi(\log n) n^{2\alpha_1 - 1}$ which proves the expansion. ∎

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# Appendix A

*Proof of Proposition 3.1.* These arguments are well-known. By construction,

$$\mathcal{Z}_{n+1,k}^{(v)} - \mathcal{Z}_{n,k}^{(v)} = \sum_{j=0}^{2^d-1} \left(\Delta_j^{(v)}\right)^{\lambda_k} \cdot \left(\mathcal{Z}_{n,k}^{(vj)} - \mathcal{Z}_{n-1,k}^{(vj)}\right),$$

and therefore

$$\Delta_n^{(v)} := \mathbb{E}\left[\left|\mathcal{Z}_{n+1,k}^{(v)} - \mathcal{Z}_{n,k}^{(v)}\right|^2\right] = \mathbb{E}\left[\left|\mathcal{Z}_{n,k}^{(v)} - \mathcal{Z}_{n-1,k}^{(v)}\right|^2\right] \sum_{j=0}^{2^d-1} \mathbb{E}\left[\left(\Delta_j^{(v)}\right)^{2\alpha_k}\right] =: q \cdot \Delta_{n-1}^{(v)}.$$

As $0 < q < 1$, it immediately follows that $\mathbb{E}\big[\big|\mathcal{Z}_{n,k}^{(v)}\big|^2\big], n \geq 1$ is a bounded sequence. Since $\mathcal{Z}_{n,k}^{(v)}, n \geq 1$ is a martingale, the sequence converges almost surely and in $L_2$ by the $L_2$-convergence theorem for martingales. This shows (6).

*(i)* and *(ii)* follow from the construction.

*(iii)* follows from a standard contraction argument for probability measures on $\mathbb{C}$ with mean $\gamma_k$ and finite second moment. Convergence of $p$-th moments is proved inductively using $p = 2$ as base case; details will be given in the journal version of this paper. ∎

*Proof of Proposition 3.2.* Leckey [12] recently established a set of conditions under which solutions of fixed-point equations such as (7) admit Schwartz densities. More precisely, since we have already seen that $\mathcal{Z}_k$ has finite moments of all orders, applying [12, Theorem 4.2] in conjunction with Remark 4.9 only requires to verify conditions (A1) - (A5) from Definition 4.1. The only condition which is not trivially satisfied is (A4): the support of $\mathcal{Z}_k$ ought to be in general position, that is, contain three points $z_1, z_2, z_3$ which do not lie on a line. For all $x \in [0, 1]$, the vector $(x, 1-x, 0, 0, \ldots, 0)$ lies in the support of $\Delta$. Therefore, $(x^{\lambda_k}, (1-x)^{\lambda_k}, 0, 0, \ldots, 0)$ lies in the support of $\Delta^{\lambda_k}$. Hence, for any $z$ in the support of $\mathcal{Z}_k$, also $(x^{\lambda_k} + (1-x)^{\lambda_k})z$ lies in the support of $\mathcal{Z}_k$. As the support of $\mathcal{Z}_k$ contains a non-zero element and $\beta_k \neq 0$, this concludes the proof. ∎

# Appendix B

*Proof of Lemma 4.3.* We start with part (a). Let $n_0$ be the first index such that $b_{n_0} > 0$. Set

$$\tilde{b}_n = \begin{cases} 0, & \text{if } 1 \leq n \leq n_0 \\ b_n + 2^d \pi_{n,n_0} b_{n_0}, & \text{if } n \geq n_0 + 1 \end{cases}$$

and denote by $\tilde{a}_n$ the corresponding sequence. Obviously, $a_n \geq \tilde{a}_n$ and thus it suffices to prove the claim for the sequence $\tilde{a}_n$. Note that by the above definition

$$\tilde{b}_n \geq c\frac{\log^d n}{n}, \qquad (n \geq n_0 + 1)$$

for some positive $c > 0$ since

$$\pi_{n,j} = \frac{1}{d!} \cdot \frac{\log^d n}{n}\left(1 + \mathcal{O}\left(\frac{1}{\log n}\right)\right)$$

for fixed $j$ (see Lemma 4 in [7]). We now claim that

$$\tilde{a}_n \geq d\left(n + \frac{1}{2^d - 1}\right), \qquad (n \geq n_0 + 1) \tag{12}$$

for some $d > 0$ which will be chosen below. We prove this claim by induction. Clearly, the claim is true for $n = n_0 + 1$. Next, in order to prove the induction step, plug the above claim into the recurrence for $\tilde{a}_n$. This yields

$$\begin{aligned}\tilde{a}_n &\geq d2^d \sum_{0\leq j<n} \pi_{n,j}\left(j+\frac{1}{2^d-1}\right) - d2^d \sum_{0\leq j<n_0+1} \pi_{n,j}\left(j+\frac{1}{2^d-1}\right) + c\frac{\log^d n}{n}\\ &\geq d\left(n-1+\frac{2^d}{2^d-1}\right) + (c-dK)\frac{\log^d n}{n}\\ &\geq d\left(n+\frac{1}{2^d-1}\right),\end{aligned}$$

where in the second estimate we used

$$2^d \sum_{0\leq j<n} j\cdot\pi_{n,j} = \mathbb{E}\left(\sum_{\ell=0}^{2^d-1} N_\ell\right) = n-1$$

and

$$2^d \sum_{0\leq j<n_0+1} \pi_{n,j}\left(j+\frac{1}{2^d-1}\right) \leq K\frac{\log^d n}{n}$$

which follows from (12). Moreover, the last estimate follows if $d$ is chosen such that $0 < d \leq c/K$. This concludes the induction step and thus also the proof.

(b) Assume that $b_n > 0$ for all $n \geq n_0$. The claim follows from part (a) by setting

$$\tilde{b}_n = \begin{cases} 0, & \text{if } 1\leq n<n_0,\\ b_n, & \text{if } n\geq n_0\end{cases}$$

and noting that the corresponding sequence $\tilde{a}_n$ satisfies $a_n \geq \tilde{a}_n$. ∎

*Proof of Lemma 4.5.* Throughout the proof, let $\alpha = \Re(z)$. Further, here, and subsequently, we write $\text{Bin}(n-1,u)$ for a random variable with binomial distribution with parameters $n-1$ and $u$.

*(i)* By construction, $\Delta_0$ is distributed as $\exp(-\Gamma^*(d))$, where $\Gamma^*(d)$ is a random variable with the Gamma distribution with density $((d-1)!)^{-1}t^{d-1}\exp(-t)$ for $t > 0$. It follows that $\Delta_0$ has density $((d-1)!)^{-1}(-\log t)^{d-1}$ for $t \in (0,1)$. Hence,

$$\left|\mathbb{E}\left[N_0^z \mathbf{1}_{[0,n^{-1+\varepsilon}]}(\Delta_0)\right]\right| \leq \mathbb{E}\left[\text{Bin}(n,n^{-1+\varepsilon})\right]^\alpha \mathbb{P}\left(\Delta_0 \leq n^{-1+\varepsilon}\right) \leq Cn^{-1+\varepsilon}\log n.$$

Next, by part *(i)* of the (well-known) postponed Lemma 4.9 below, we have

$$\begin{aligned}\mathbb{E}\left[N_0^z \mathbf{1}_{[n^{-1+\varepsilon},1]}(\Delta_0)\right] &= \mathbb{E}\left[\Delta_0^z \mathbf{1}_{[n^{-1+\varepsilon},1]}(\Delta_0)\right] n^z\\ &\quad + \frac{z(z-1)}{2}\mathbb{E}\left[(1-\Delta_0)\,\Delta_0^{z-1}\mathbf{1}_{[n^{-1+\varepsilon},1]}(\Delta_0)\right] n^{z-1} + \mathcal{O}(n^{\varepsilon-1}).\end{aligned}$$

Dropping the indicators on the right hand side only adds a negligible error term as $\left|\mathbb{E}\left[\Delta_0^z \mathbf{1}_{[0,n^{-1+\varepsilon})}(\Delta_0)\right]\right| \leq n^{(-1+\varepsilon)\Re(z)-1+\varepsilon}\log n$ with a similar computation for the second summand.

*(ii)* We have

$$\begin{aligned}\mathbb{E}\left[|N_0^z - (\Delta_0 n)^z|^p\right] &\leq 2^p((d-1)!)^{-1}(\mathbb{E}\left[\text{Bin}(n-1,1/n)^{\alpha p}\right]+1)\int_0^{1/n}(-\log u)^{d-1}du\\ &\quad + ((d-1)!)^{-1}\int_{1/n}^1 (-\log u)^{d-1}\mathbb{E}\left[|(\text{Bin}(n-1,u))^z-(un)^z|^p\right]du.\end{aligned}$$

Part *(ii)* of Lemma 4.9 below shows that the integral in the second summand is bounded by

$$C\int_{1/n}^{\infty}(-\log u)^{d-1}\left((un)^{p(\alpha-1/2)}+(un)^{p\alpha}e^{-Cun}\right)du=\mathcal{O}(n^{p(\alpha-1/2)}).$$

As

$$\int_0^{1/n}(-\log u)^{d-1}du=\frac{1}{n}(\log n)^{d-1}(1+\mathcal{O}((\log n)^{-1})),$$

it follows that

$$\|N_0^z-(\Delta_0 n)^z\|_p=\mathcal{O}\left(n^{\alpha-1/2}\right).$$

This shows that the marginals in the mutivariate central limit theorem stated in Lemma 4.4 converge with respect to all moments. This shows *(ii)*. *(iii)* follows along similar lines. ∎

**Lemma 4.9.** *Let $z\in\mathbb{C}$ with $0<\alpha:=\Re(z)<1$. We have the following asymptotic expansions:*

*(i) for any $\varepsilon>0$ sufficiently small, as $n\to\infty$, uniformly in $n^{-1+\varepsilon}\le u\le 1$,*

$$\mathbb{E}[\mathrm{Bin}(n,u)^z]=(nu)^z+\frac{z(z-1)}{2}(1-u)(nu)^{z-1}+\mathcal{O}(n^{\varepsilon-1}).$$

*(ii) For $p\in\mathbb{N}\setminus\{0\}$, there exists a constant $C>0$ such that*

$$\mathbb{E}\left[|\mathrm{Bin}(n,u)^z-(nu)^z|^p\right]\le C\left(\mathbf{1}_{(0,1/n)}(u)+\mathbf{1}_{[1/n,1]}(u)\left((un)^{p(\alpha-1/2)}+(un)^{p\alpha}e^{-Cun}\right)\right).$$

*Proof.* *(i)* On $[1/2,3/2]$, we have

$$x^z=1+z(x-1)+\frac{z(z-1)}{2}(x-1)^2+\gamma(x)(x-1)^3,$$

for some function $\gamma$ which is bounded on $[1/2,3/2]$. Let $A=\{\mathrm{Bin}(n,u)/(nu)\in[1/2,3/2]\}$. Plugging $x=\mathrm{Bin}(n,u)\mathbf{1}_A/(nu)$ into the last display and taking the expectation gives

$$\begin{aligned}\frac{\mathbb{E}[\mathrm{Bin}(n,u)^z\mathbf{1}_A]}{(nu)^z}&=1+z\mathbb{E}\left[\frac{\mathrm{Bin}(n,u)\mathbf{1}_A}{nu}-1\right]+\frac{z(z-1)}{2}\mathbb{E}\left[\left(\frac{\mathrm{Bin}(n,u)\mathbf{1}_A}{nu}-1\right)^2\right]\\&\quad+\mathcal{O}\left(\mathbb{E}\left[\left(\frac{\mathrm{Bin}(n,u)\mathbf{1}_A}{nu}-1\right)^3\right]\right).\end{aligned}$$

By Chernoff's inequality, since $u\ge n^{-1+\varepsilon}$, we have $\mathbb{P}(A)\le C_1\exp(-C_2n^{\varepsilon})$ for some universal constants $C_1,C_2>0$. Hence, dropping the indicator $\mathbf{1}_A$ in all expectations in the last display adds a negligible error term.

*(ii)* For $u\le 1/n$, we can bound $\mathbb{E}\left[\mathrm{Bin}(n,u)^{\alpha k}\right]\le\mathbb{E}\left[\mathrm{Bin}(n,1/n)^{\alpha k}\right]\to\mathbb{E}\left[P^{\alpha k}\right]$ as $n\to\infty$. (Here, $P$ denotes a random variable with the Poisson distribution and mean one.) Obviously, $(nu)^{\alpha k}\le 1$. This shows one part of the inequality. For the more interesting case $u\ge 1/n$, first observe that

$$\begin{aligned}\mathbb{E}\left[|\mathrm{Bin}(n,u)^z-(nu)^z|^p\right]&\le 2^k\left(\mathbb{E}\left[|\mathrm{Bin}(n,u)^\alpha-(nu)^\alpha|^p\right]+\mathbb{E}\left[\left|\mathrm{Bin}(n,u)^\alpha\cdot\log\frac{\mathrm{Bin}(n,u)}{nu}\right|^p\right]\right)\\&=:2^k(f_1(u,n)+f_2(u,n)).\end{aligned}$$

Set $E_n=\{\mathrm{Bin}(n,u)>(nu)/2\}$ and define

$$\begin{aligned}f_1(u,n)&=\mathbb{E}\left[|\mathrm{Bin}(n,u)^\alpha-(nu)^\alpha|^p\,\mathbf{1}_{E_n}\right]+\mathbb{E}\left[|\mathrm{Bin}(n,u)^\alpha-(nu)^\alpha|^p\,\mathbf{1}_{E_n^c}\right]\\&=:g_1(u,n)+h_1(u,n),\end{aligned}$$

and

$$f_1(u,n) = \mathbb{E}\left[\left|\text{Bin}(n,u)^\alpha \cdot \log\frac{\text{Bin}(n,u)}{nu}\right|^p \mathbf{1}_{E_n}\right] + \mathbb{E}\left[\left|\text{Bin}(n,u)^\alpha \cdot \log\frac{\text{Bin}(n,u)}{nu}\right|^p \mathbf{1}_{E_n^c}\right]$$
$$=: g_2(u,n) + h_2(u,n),$$

We now give bounds on $g_1, g_2, h_1$ and $h_2$. Let $\varrho(t) = (1+t)^\alpha$. Then, $|\varrho'(t)| \le \alpha 2^{1-\alpha}$ for all $t \ge -1/2$. Thus, by the postponed Lemma 4.10 below,

$$\begin{aligned} g_1(u,n) &= (nu)^{\alpha p}\mathbb{E}\left[\left|\varrho\left(\frac{\text{Bin}(n,u)-nu}{nu}\right) - 1\right|^p \mathbf{1}_{E_n}\right] \\ &\le (\alpha 2^{1-\alpha})^p (nu)^{p(\alpha-1)}\mathbb{E}\left[|\text{Bin}(n,u)-nu|^p\right] \le C(un)^{p(\alpha-1/2)} \end{aligned}$$

for some $C > 0$. Next, we consider $g_2$. Let $\psi(t) = t^\alpha |\log t|$. As $\psi'$ is bounded on $[1/2,\infty)$, by, say $C_1 > 0$, we have

$$\begin{aligned} g_2(u,n) &= (nu)^{\alpha p}\mathbb{E}\left[\psi\left(\frac{\text{Bin}(n,u)-nu}{nu}\right)^p \mathbf{1}_{E_n}\right] \\ &\le C_1 (nu)^{p(\alpha-1)}\mathbb{E}\left[|\text{Bin}(n,u)-nu|^p\right] \le C_2(un)^{p(\alpha-1/2)}. \end{aligned}$$

for some $C_2 > 0$. Next, by the Cauchy-Schwarz inequality and the postponed Lemma 4.10 below,

$$\begin{aligned} h_1(u,n) &\le \mathbb{E}\left[|\text{Bin}(n,u)-nu|^{2p\alpha}\right]^{1/2} \mathbb{P}\left(E_n^c\right)^{1/2} \\ &\le C_1 (nu)^{p\alpha/2} e^{-Cun} \end{aligned}$$

for some $C > 0$. Since $\psi$ in bounded on $[0,1]$ by, say $C > 0$, we also have

$$h_2(u,n) \le (nu)^{\alpha p}\mathbb{E}\left[\psi\left(\frac{\text{Bin}(n,u)}{nu}\right)^p \mathbf{1}_{E_n^c}\right] \le (nu)^{\alpha p} e^{-Cun}.$$

This concludes the proof. ∎

**Lemma 4.10.** *For any real $r \ge 1$ there exists a constant $C > 0$ such that, for all $n \ge 1$ and $u \in [0,1]$, we have*

$$\mathbb{E}\left[|\text{Bin}(n,u)-nu|^r\right] \le C(nu)^{r/2}.$$

*Proof.* By Jensen's inequality, we may restrict ourselves to the case of integer $r$. Using Bernstein's inequality, we obtain

$$\begin{aligned} \mathbb{E}\left[|\text{Bin}(n,u)-nu|^r\right] &= r\int_0^\infty y^{r-1}\mathbb{P}\left(|\text{Bin}(n,u)-nu| \ge y\right) dy \\ &\le r\int_0^\infty y^{r-1}\exp\left(-\frac{y^2}{2nu+2y/3}\right) dy \\ &\le r\int_0^{6nu} y^{r-1}\exp\left(-\frac{y^2}{6np}\right) dy + r\int_{6nu}^\infty y^{r-1}e^{-y} dy. \end{aligned}$$

Sustituting $x = y/\sqrt{6nu}$, one finds that the first term is bounded by $C(nu)^{k/2}$ for all $n \ge 1$ and $u \in [0,1]$. The second summand is $\mathcal{O}(\exp(-\alpha np))$ for any $\alpha < 6$. ∎