Phase changes in random point quadtrees

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Dedicated to the memory of Ching-Zong Wei (1949–2004)

Abstract

We show that a wide class of linear cost measures (such as the number of leaves) in random d-dimensional point quadtrees undergo a change in limit laws: if the dimension $d = 1, \dots, 8$, then the limit law is normal; if $d \geq 9$ then there is no convergence to a fixed limit law. Stronger approximation results such as convergence rates and local limit theorems are also derived for the number of leaves, additional phase changes being unveiled. Our approach is new and very general, and also applicable to other classes of search trees. A brief discussion of Devroye's grid-trees (covering m-ary search trees and quadtrees as special cases) is given. We also propose an efficient numeric procedure for computing the constants involved to high precision.

1 Introduction

Phase transitions in random combinatorial objects issuing from computer algorithms have received much recent attention by computer scientists, probabilists, and statistical physists, especially for NP-complete problems. We address in this paper the change of the limit laws from normal to non-convergence of some cost measures in random point quadtrees when the dimension varies. The phase change phenomena[1], as well as the asymptotic tools we develop (based mostly on linear operators), are of some generality. We will discuss the corresponding phase changes in Devroye's random grid-trees (see [12]) for which a complete description of the phase changes will be given.

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[1]We use mostly "phase change" instead of "phase transition" because the dimension in our problem takes only positive integers.

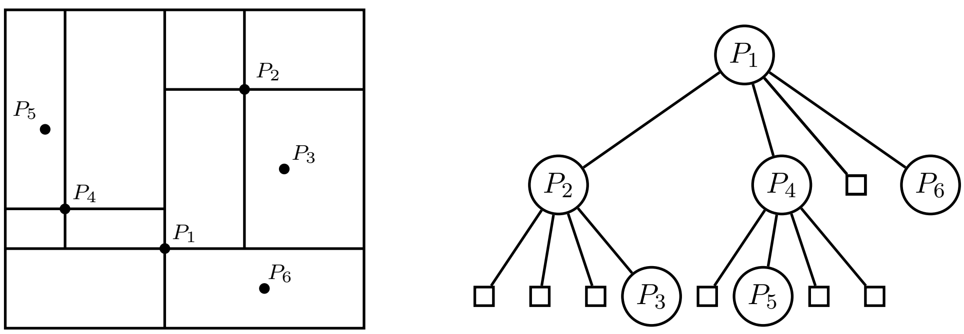

Figure 1: *A configuration of* 6 *points in the unit square and the corresponding quadtree.*

Point quadtrees. Point quadtrees, first introduced by Finkel and Bentley [16], are useful spatial and indexing data structures in computational geometry and for low-dimensional points in diverse applications in practice; see de Berg et al. [9], Samet [43, 44] for more information. In this paper, *we will say quadtrees instead of point quadtrees* for simplicity.

Given a sequence of points in $\mathbb{R}^d$, the quadtree associated with this point sequence is constructed as follows. The first point is placed at the root and then splits the underlying space into 2^d smaller regions (or quadrants), each corresponding to one of the 2^d subtrees of the root. The remaining points are directed to the quadrants (or the corresponding subtrees), and the subtrees are then constructed recursively by the same procedure. See Figure 1 for a plot of $d = 2$. When $d = 1$, quadtrees are simply binary search trees. Thus quadtrees can be viewed as one of the many different extensions of binary search trees; see [7, 12, 37].

Random quadtrees. To study the typical shapes or cost measures of quadtrees, we assume that the given points are uniformly and independently chosen from $[0, 1]^d$, where $d \geq 1$, and then construct the quadtree associated with the random sequence; the resulting quadtree is called a *random quadtree*.

Several shape parameters and cost measures in random quadtrees have been studied, reflecting in different levels certain typical complexity of algorithms on quadtrees.

- Depth (distance of a randomly chosen node to the root): [12, 13, 17, 19, 20];
- Total path length (sum of distances of all nodes to the root): [17, 19, 40];
- Cost of partial-match queries: [4, 17, 38, 41];
- Node types: [19, 26, 34, 35, 36];
- Height (distance of the longest path to the root): [10, 12].

In particular, the asymptotic normality of the depth was first proved in Flajolet and Lafforgue [20] (see also [12]), and the non-normal limit law for the total path length in Neininger and Rüschendorf [40].

The number of leaves. For concreteness and simplicity, we present the phase change phenomena through the number of leaves, denoted by $X_n = X_{n,d}$, in random quadtrees of n points. The extension to more general cost measures will be discussed later.

When $d = 1$, it is known that X_n (the number of leaves in random binary search trees of n nodes) is asymptotically normally distributed with mean and variance asymptotic to $n/3$ and $2n/45$, respectively; see [11, 18]. A local limit theorem is also given in [18].

For $d \geq 2$, Flajolet et al. (see [19]) first derived the closed-form expression for the expected value of X_n

$$\mathbb{E}(X_n) = n - \sum_{2\leq k\leq n} \binom{n}{k} (-1)^k [k]! \sum_{2\leq j\leq k} \frac{1}{[j]!} \qquad (n \geq 1), \tag{1}$$

where $[k]! := \prod_{3\leq j\leq k}(1 - 2^d/j^d)$ for $k \geq 3$ and $[2]! := 1$, and then showed that

$$\mathbb{E}(X_n) \sim \mu_d n,$$

where

$$\mu_d := 1 - \frac{2}{d} \prod_{\ell\geq 3} \frac{1}{1 - \left(\frac{2}{\ell}\right)^d} + 2^{d+1} \sum_{j\geq 2} \frac{1}{[j]!} \sum_{h\geq 1} \frac{1}{(h+j)((h+j)^d - 2^d)}; \tag{2}$$

see (50) for an alternative expression. In particular, $\mu_1 = 1/3$ and $\mu_2 = 4\pi^2 - 39$; see [26, 36].

The phase change. Our first result says that when d increases, there is a change of nature for the limit distribution of X_n.

Theorem 1. (i) *If* $1 \leq d \leq 8$, *then*

$$\frac{X_n - \mu_d n}{\sigma_d \sqrt{n}} \xrightarrow{\mathscr{M}} N(0, 1),$$

where $\xrightarrow{\mathscr{M}}$ *denotes convergence of all moments and* $N(0, 1)$ *is the standard normal random variable (zero mean and unit variance). The constants* σ_d *are given in (52).*

(ii) *If* $d \geq 9$, *then the sequence of random variables* $(X_n - \mathbb{E}(X_n))/\sqrt{\mathbb{V}(X_n)}$ *does not converge to a fixed limit law.*

In the first case, convergence in distribution of $(X_n - \mu_d n)/\sqrt{\sigma_d^2 n}$ is also implied.

Why phase change? One key (analytic) reason why the limiting behavior of X_n changes its nature for $d \geq 9$ is because of the second order term in the asymptotic expansion of $\mathbb{E}(X_n)$

$$\mathbb{E}(X_n) = \mu_d n + G_1(\beta \log n)n^\alpha + o(n^\alpha + n^\varepsilon) \qquad (d \geq 2), \tag{3}$$

where $\alpha := 2\cos(2\pi/d) - 1$, $\beta := 2\sin(2\pi/d)$, and $G_1(x)$ is a bounded, 1-periodic function; see (49) for an explicit expression. If $d \leq 8$, then $\alpha < 1/2$; and $\alpha \in (1/2, 1)$ if $d \geq 9$; see Table 1 for numeric values of α.

d	2	3	4	5	6	7	8	9
α	-3	-2	-1	-0.38	0	0.24	0.41	0.53

Table 1: *Approximate numeric values of* $\alpha = 2\cos(2\pi/d) - 1$ *for* d *from* 2 *to* 9.

From this expansion, we can derive the asymptotics of the variance

$$\mathbb{V}(X_n) \sim \begin{cases} \sigma_d^2 n, & \text{if } 1 \le d \le 8; \\ G_2(\beta \log n) n^{2\alpha}, & \text{if } d \ge 9, \end{cases} \tag{4}$$

where $G_2(x)$ is a bounded, 1-periodic function.

Intuitively, we see that the periodicity in (3) becomes more pronounced as d grows (see Figure 2), implying larger and larger variance in (4), so that in the end $(X_n - \mathbb{E}(X_n))/\sqrt{\mathbb{V}(X_n)}$ does not converge to a fixed limit law.

Phase changes in other search trees. The situation here is similar to several phase change phenomena already studied in the literature in many varieties of random search trees and related algorithms: m-ary search trees, fringe-balanced binary search trees, generalized quicksort, etc; see [2, 3, 7, 15, 28, 29]. See also Janson [33] for a very complete description of phase changes in urn models, which are closely connected to many random search trees.

However, the analytic context here is much more involved than previously studied search trees because, as we will see, the underlying differential equation is no more of Cauchy-Euler type, which demands more delicate analysis.

Phase changes in random fragmentation models. The same phase change phenomenon as leaves in random quadtrees was first observed in Dean and Majumdar [8], where they proposed *random continuous fragmentation models* to *explain heuristically* the phase changes in random search trees. Their continuous model corresponding to quadtrees is as follows. Pick a point in $[0, x]^d$ uniformly at random ($x \gg 1$), which then splits the space into 2^d smaller hyperrectangles. Continue the same procedure in the sub-hyperrectangles whose volumes are larger than unity. The process stops when all sub-hyperrectangles have volumes less than unity. They argue heuristically that the total number of splittings undergoes a phase change: "While we can rigorously prove that the distribution is indeed Gaussian in the sub-critical regime [$d \le 8$], we have not been able to calculate the full distribution in the super-critical regime [$d \ge 9$]"; see [8].

Recently, Janson (private communication) showed that the same type of phase change can be constructed by considering the number of nodes at distance ℓ satisfying $\ell \bmod d \equiv j$, $0 \le j < d$, in random binary search trees, or equivalently, the number of nodes using the $(\ell+1)$-st coordinate as discriminators in random k-d trees, where $\ell \bmod d \equiv j$.

Recurrence. By the recursive nature of the problem proper, X_n satisfies the recurrence

$$X_n \stackrel{\mathscr{D}}{=} X_{J_1}^{(1)} + \cdots + X_{J_{2^d}}^{(2^d)} + \delta_{n,1} \qquad (n \ge 1), \tag{5}$$

with $X_0 = 0$, where the symbol $\stackrel{\mathscr{D}}{=}$ denotes equality in distribution, the J_i's and the $X_n^{(i)} \stackrel{\mathscr{D}}{=} X_n$'s are independent, $\delta_{n,1}$ denotes the Kronecker symbol, and

$$\begin{aligned} \pi_{n,\mathbf{j}} &:= \mathbb{P}(J_1 = j_1, \cdots J_{2^d} = j_{2^d}) \\ &= \binom{n-1}{j_1, \ldots, j_{2^d}} \int_{[0,1]^d} q_1(\mathbf{x})^{j_1} \cdots q_{2^d}(\mathbf{x})^{j_{2^d}} \, \mathrm{d}\mathbf{x}, \end{aligned}$$

denotes the probability that the 2^d subtrees of the root are of sizes $j_1, \dots, j_{2^d}$. Here $\mathrm{d}\mathbf{x} = \mathrm{d}x_1 \cdots \mathrm{d}x_d$ amd the $q_i(\mathbf{x})$'s denote the volumes of the hyperrectangles split by a random point $\mathbf{x} = (x_1, \dots, x_d)$. We can arrange the $q_i(\mathbf{x})$'s as follows

$$q_h(\mathbf{x}) = \prod_{1 \le i \le d} \left((1-b_i)x_i + b_i(1-x_i)\right) \qquad (1 \le h \le 2^d), \tag{6}$$

where $(b_1, \dots, b_d)_2$ stands for the binary representation of $h-1$ (the first few digits being completed with zeros if $\lfloor \log_2(h-1) \rfloor < d-1$, so that $0 = (\underbrace{0, \dots, 0}_{d})_2$, $1 = (\underbrace{0, \dots, 0}_{d-1}, 1)_2$, etc.).

The moment-transfer approach. By (5), all moments of X_n (centered or not) satisfy the same recurrences of the form

$$A_n = B_n + 2^d \sum_{0 \le j < n} \pi_{n,j} A_j \qquad (n \ge 1), \tag{7}$$

with A_0 and $\{B_n\}_{n \ge 1}$ given, where

$$\pi_{n,j} = \binom{n-1}{j} \int_{[0,1]^d} (x_1 \cdots x_d)^j \, (1 - x_1 \cdots x_d)^{n-1-j} \, \mathrm{d}\mathbf{x}. \tag{8}$$

Many different expressions for $\pi_{n,j}$ can be found in [19, 34]; see also [25].

To prove the limit distribution, we apply the *moment-transfer approach*, which has proved successful in diverse problems of recursive nature. We have applied the approach to and developed the required asymptotic tools for many problems, including m-ary search trees, generalized quicksort and most variations of quicksort, bucket digital search trees, maximum-finding algorithms in distributed networks, maxima in right triangle; see the survey paper [29] for more references.

The basic idea of the approach is, because all moments satisfy the same recurrence (7), to incorporate the analysis of the asymptotics of higher moments into developing the so-called *asymptotic transfer*, which, roughly speaking, infers asymptotics of A_n from that of B_n. Such an approach always reduces most analysis to obtaining the first or second moments, the remaining part being more or less mechanical. It also offers the possibility of refining the limit theorems by stronger approximation results like convergence rates and local limit theorems, the new ingredients needed being developed in [28] for m-ary search trees; see also [1].

Second phase change. The refined moment-transfer approach (see [28]) shows that X_n undergoes a second phase change in convergence rate to normal limit law (often referred to as the Berry-Esseen bound). Our result says that the convergence rate to normal law is of order $n^{-1/2}$ when $1 \le d \le 7$, but is of a poorer order $n^{-3(3/2-\sqrt{2})} \approx n^{-0.24}$ when $d = 8$. Both rates are optimal modulo the implied constants. We will indeed derive local limit theorems for X_n, which are more precise and informative than convergence in distribution.

Resolution of the recurrence (7). *Exact solutions* of the recurrence (7) were first investigated by Flajolet et al. in [19] (see also [36, 39]), based mainly on the crucial introduction of the Euler transform. *Asymptotic properties* of (7) were also thoroughly examined in [19], using powerful complex-analytic tools. Their approach is very efficient in deriving the asymptotic expansions, but requires stronger information on the given "toll sequence" B_n.

In this paper, we show that the exact solution given via Euler transform in [19] (see (19)) can also be obtained by using the usual Poisson generating functions. Although this approach is essentially the same as the Euler transform on ordinary generating functions, it offers an operational advantage in simplifying the calculation of the exact variance; see Section 3.2.

Asymptotic transfer of the recurrence (7). We will develop the asymptotic transfer needed for deriving asymptotics of moments. Most proofs of previously known phase changes in random search trees and quicksort algorithms rely more or less on developing the asymptotic transfer for Cauchy-Euler differential equations (abbreviated as DEs) of the form

$$\text{Polynomial}(\vartheta)\xi(z) = \eta(z), \tag{9}$$

where η is independent of ξ and $\vartheta := (1-z)(\mathrm{d}/\mathrm{d}z)$. The main transfer problem under this framework is to derive asymptotics of $[z^n]\xi(z)$ when that of $[z^n]\eta(z)$ is known, where $[z^n]\xi(z)$ denotes the coefficient of z^n in the Taylor expansion of f. A very general, elementary asymptotic theory for such DEs with a large number of applications is given in [7], the origin of such a development being traceable to Sedgewick's analysis on quicksort (see [45]).

For quadtrees, the DE satisfied by the generating function $A(z) := \sum_n A_n z^n$ is given by

$$\vartheta(z\vartheta)^{d-1}(A(z) - B(z)) = 2^d A(z), \tag{10}$$

which is not of the type (9) but can be rewritten in the extended form

$$P_0(\vartheta)A(z) = \vartheta(z\vartheta)^{d-1}B(z) + \sum_{1\le j<d} (1-z)^j P_j(\vartheta)A(z), \tag{11}$$

where $P_0(x) = x^d - 2^d$ and the $P_j(x)$'s are polynomials of degree d; see (23).

We then extend the iterative operator approach introduced in [5] to analyzing the expected cost of partial match queries in random k-d trees. The approach turns out to be very useful for extended Cauchy-Euler DEs of the form (11); see [6] for another application to consecutive records in random sequences.

The main differences of the current application from the previous ones are: (i) we consider general non-homogeneous part (or toll functions) rather than specific ones; (ii) the method of Frobenius (and the method of annihilators) used in our previous papers is avoided and replaced by a more uniform elementary argument, the resulting proof being completely elementary and requiring almost no knowledge on DE; (iii) we give not only necessary but also sufficient conditions for all transfers we developed; the same proof for the sufficiency part also easily modified for proving the necessity in all cases, keeping uniformity of the approach; (iv) the proof we give in its current form is easily amended for more general DEs with polynomial coefficients; (v) we put forth means of simplifying the expressions for the constants involved; the resulting expressions are in some cases simpler than those derived in [19]; also our expressions are easily amended for numeric purposes.

A universal condition for asymptotic linearity? One main result our approach can achieve states that A_n is asymptotically linear $A_n \sim Kn$ if and only if $B_n = o(n)$ and the series $\sum_n B_n n^{-2}$ is convergent, where K is explicitly given in terms of the B_n's; see (16). It is interesting to see that exactly the same condition for the asymptotic linearity of A_n holds for other recurrences appearing in quicksort, m-ary search trees, generalized quicksort, and many others; see [7]. Note that the expression for the linearity constant K differs from one case to another. The series condition $|\sum_n B_n n^{-2}| < \infty$ also arises in many other problems such as generalized subadditive inequalities, divide-and-conquer algorithms, large deviations, etc.; see [31] and the references therein. Is there a deeper reason why the series condition is so universal?

Organization of the paper. In the next Section 2, we develop general asymptotic transfer results, which can be applied to more general shape characteristics and cost measures. In Sections 3 and 4, we study the phase change phenomena exhibited by the number of leaves and discuss the extension to general cost measures. Effective numerical procedures will also be given of computing the limiting mean and variance constants for X_n. The extension of our consideration to Devroye's grid-trees (see [12]) is given in the final section.

Notation. Throughout this paper, the notation $[z^n]f(z)$ denotes the coefficient of z^n in the Taylor expansion of f. The generic symbol ε always represents some small quantity whose value may vary from one occurrence to another; similarly, the generic symbol c stands for a suitable constant. We define two operators $\mathbb{D}_z := \mathrm{d}/\mathrm{d}z$ and $\vartheta := (1-z)\mathbb{D}_z$. The same set of symbols $\{B_n, B(z), B^*(s)\}$ is used for the sequence B_n, its generating function $B(z) = \sum_n B_n z^n$, and its factorial series or Mellin transform $B^*(s) = \int_0^1 (1-x)^{s-1} B(x)\,\mathrm{d}x$, respectively.

2 Asymptotic transfer of the quadtree recurrence

We develop the asymptotic tools in this section by proving the different types of asymptotic transfer needed for later uses. A salient feature of our transfers is that the asymptotic condition in each case is not only sufficient but also proved to be necessary.

Three types of asymptotic transfer. For simplicity, we assume $A_0 = 0$ since otherwise the difference is given explicitly by $A_0(2^d-1)n + A_0$; see (19).

Theorem 2. *Let A_n be defined by the recurrence (7) with A_0 and $\{B_n\}_{n\ge1}$ given. Then*

(i) *(Small toll functions)*

$$A_n \sim K_B n \quad \text{iff} \quad B_n = o(n) \ \text{ and } \ \left|\sum_n B_n n^{-2}\right| < \infty, \tag{12}$$

where the constant K_B is given in (16);

(ii) *(Linear toll functions) Assume that $B_n = cn + u_n$, where $c \in \mathbb{C}$ and u_n is a sequence of complex numbers. Then*

$$A_n \sim \frac{2}{d} cn\log n + K_1 n \quad \text{iff} \quad u_n = o(n) \text{ and } \left|\sum_n u_n n^{-2}\right| < \infty, \tag{13}$$

where $K_1 := cK_2 + K_u$ with K_u defined by replacing the sequence B_n by u_n in (16) and K_2 given explicitly by

$$K_2 := -1 - \frac{2}{d} + 2\gamma + \frac{2}{d}\sum_{1\le j<d} \psi(2 - 2e^{2j\pi i/d}), \tag{14}$$

ψ being the logarithmic derivative of the Gamma function (see [14]);

(iii) *(Large toll functions) Assume that $\Re(\upsilon) > 1$ and $c \in \mathbb{C}$. Then*

$$B_n \sim cn^{\upsilon} \quad \text{iff} \quad A_n \sim \frac{c(\upsilon+1)^d}{(\upsilon+1)^d - 2^d} n^{\upsilon}. \tag{15}$$

More refinements to (12) under stronger assumptions on B_n will be proved below.

The linearity constant. Given a sequence B_n, define the constant K_B by the series

$$K_B = \frac{2}{d} \sum_{k\ge 0} V_k B^*(k+2), \tag{16}$$

which is absolutely convergent under the condition (12) on B_n, where V_k is defined recursively by $V_k = 0$ when $k < 0$, $V_0 = 1$, and

$$V_k = \sum_{1\le \ell<d} \frac{P_\ell(k+2)}{P_0(k+2)} V_{k-\ell} \qquad (k \ge 1), \tag{17}$$

and the function B^* is given by

$$B^*(s) := \int_0^1 B(x)(1-x)^{s-1}\,\mathrm{d}x = \sum_{j\ge 1} \frac{B_j j!}{s(s+1)\cdots(s+j)}, \tag{18}$$

when the integral and series converge. Here the polynomials $P_j(x)$'s are given in (23). Note that when $d = 1$, $V_k = \delta_{k,0}$, so that $K_B = 2B^*(2)$; see [30].

2.1 Euler transform and Poissonization

Euler transform. Flajolet et al. proposed in [19] an approach via Euler transform for solving the recurrence (7); their result is

$$A_n = A_0 + n\left((2^d-1)A_0 + B_1\right) + \sum_{2\le k\le n} \binom{n}{k}(-1)^k \sum_{2\le j\le k} \left(B_j^\star - B_{j-1}^\star\right) \prod_{j<\ell\le k} \left(1 - \frac{2^d}{\ell^d}\right), \tag{19}$$

for $n \ge 0$, where $B_n^\star$ denotes the Euler transform of the sequence B_n

$$B_n^\star := \sum_{1\le j\le n} \binom{n}{j}(-1)^j B_j.$$

As one can see from (19), the appearance of $B_n^\star$ and the power of -1 makes the asymptotics of A_n less transparent.

Poissonization. An alternative way of deriving (19) is as follows. Consider the Poisson generating functions of both sequences: $\tilde{A}(z) := e^{-z}\sum_{n\ge 0} A_n z^n/n!$ and $\tilde{B}(z) := e^{-z}\sum_{n\ge 1} B_n z^n/n!$. Then (7) translates into

$$\tilde{A}'(z) + \tilde{A}(z) = \tilde{B}'(z) + \tilde{B}(z) + 2^d \int_{[0,1]^d} \tilde{A}(x_1\cdots x_d z)\,\mathrm{d}\mathbf{x},$$

with the initial condition $\tilde{A}(0) = A_0$. Let $\tilde{A}_n := n![z^n]\tilde{A}(z)$ and $\tilde{B}_n := n![z^n]\tilde{B}(z)$. Then

$$\tilde{A}_n + \tilde{A}_{n-1} = \tilde{B}_n + \tilde{B}_{n-1} + \frac{2^d}{n^d}\tilde{A}_{n-1} \qquad (n \ge 1), \tag{20}$$

(for convenience, defining $B_0 = \tilde{B}_0 = 0$). Observe that

$$\tilde{A}_n = (-1)^n A_n^\star = (-1)^n \sum_{0\le k\le n} \binom{n}{k}(-1)^k A_k,$$

and $\tilde{B}_n = (-1)^n B_n^\star$. By iterating the recurrence (20) and by taking into account the initial values, we obtain (19).

Although the approach is essentially the same as that via Euler transform, it is helpful in deriving a dimension-free expression for, say the variance of X_n; see Section 3.2. It also offers the possibility of obtaining the asymptotics of A_n by the usual Mellin transform techniques.

Asymptotics of the recurrence (7). A very powerful complex-analytic approach is proposed in [19] to the asymptotics of (7). The main idea is to apply singularity analysis (see [21]); so one needs the asymptotics of the generating function $\sum_n A_n z^n$ for $z \sim 1$, which, by the Euler transform, leads to the study of the generating function $A^\star(t) := \sum_n A_n^\star t^n$ for t near $-\infty$. For that purpose, they apply integral representation for $A^\star(-t)$ of the form

$$A^\star(-t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi t^s}{\sin \pi s}\,\varphi(s)\,\mathrm{d}s,$$

for suitably chosen c and $\varphi(s)$ satisfying $\varphi(k) = A_k^\star$ for $k \geq 2$. The determination of such an "analytic extrapolation" of $A_k^\star$ to complex s is crucial.

The major limitation of this approach is that when the given sequence B_n is, say only known up to $O(n^\alpha)$ or $\sim n^\alpha$ for some α, it is not obvious how to find an analytic extrapolation and then to deduce the right order of A_n because of the underlying "exponential cancellations of order": roughly, $\binom{n}{k}$ has its largest term of order $2^n n^{-1/2}$, but most of our sequences grow only polynomially in n; see [23] for asymptotics on alternating binomial sums.

Alternatively, one might try the usual Mellin analysis for $\tilde{A}(z)$ (or its truncated functions); again analytic properties of the involved function at $\sigma \pm i\infty$ may be very challenging.

Note that the value A_0 and the sequence $\{B_n\}_{n\geq 1}$ are enough to completely determine the sequence A_n. This property will be useful in our numeric procedure; see Section 3.2.

2.2 Asymptotic transfer I. Small toll functions

We prove the first case of Theorem 2 in this section by extending the approach we proposed before for the analysis of k-d trees. The main idea is to write the underlying DE in the form of certain "perturbed" DE of Cauchy-Euler type, and then to use some iterative operator arguments.

The DE. Let $A(z) = \sum_{n\geq 0} A_n z^n$ and $B(z) = \sum_{n\geq 1} B_n z^n$. Then the recurrence (7) translates into the DE (10), which becomes simpler by considering $f := A - B$:

$$\left(\vartheta(z\vartheta)^{d-1} - 2^d\right) f(z) = 2^d B(z). \tag{21}$$

This DE can be re-written as the "perturbed" Cauchy-Euler DE

$$\begin{cases} P_0(\vartheta)f(z) = g(z) + 2^d B(z); \\ g(z) := \sum_{1\leq j<d}(1-z)^j P_j(\vartheta)f(z), \end{cases} \tag{22}$$

where $P_0(x) = x^d - 2^d$, and by induction

$$P_j(x) = (-1)^{j-1}[z^{d-1-j}] \prod_{0\leq r\leq j} \frac{x-r}{1-z(x-r)} \qquad (1 \leq j < d). \tag{23}$$

Note that all P_j's are polynomials of degree d; they can also be computed recursively as follows. Write

$$\vartheta(z\vartheta)^{d-1}f(z) = \sum_{0\le j<d}(1-z)^j\tilde{P}_{d,j}(\vartheta)f(z).$$

Then $P_j(x) = -\tilde{P}_{d,j}(x)$ for $1 \le j < d$. Here $\tilde{P}_{d,j}(x) = (x-j)(\tilde{P}_{d-1,j}(x) - \tilde{P}_{d-1,j-1}(x))$ with the boundary conditions $\tilde{P}_{1,0}(x) = x$, $\tilde{P}_{d,j}(x) = 0$ if $j < 0$ or $j \ge d$.

Let λ_j's denote the zeroes of $P_0(x) = 0$, namely, $\lambda_j = 2e^{2j\pi i/d}$ for $0 \le j < d$. In particular, $\lambda_0 = 2$.

All initial conditions zero. For convenience, we assume temporarily that all initial values are zeros $f^{(j)}(0) = 0$ for $0 \le j < d$. This implies that $\vartheta^j f(0) = 0$ for $0 \le j < d$ since

$$\vartheta^j f(z) = \sum_{0\le \ell\le j}(-1)^{j+\ell}S(j,\ell)(1-z)^\ell f^{(\ell)}(z),$$

where $S(j,\ell)$ represents the Stirling numbers of the second kind.

The Cauchy-Euler solution. Regarding the DE (22) as a Cauchy-Euler DE, we can then decompose the DE as follows.

$$(\vartheta-\lambda_{d-1})\cdots(\vartheta-\lambda_1)(\vartheta-2)f(z) = g(z) + 2^d B(z), \tag{24}$$

whose solution (exact or asymptotic) can be obtained by successively solving the first-order DE of the form

$$(\vartheta-\upsilon)\xi(z) = \eta(z),$$

which is given by

$$\xi(z) = \xi(0)(1-z)^{-\upsilon} + (1-z)^{-\upsilon}\int_0^z (1-t)^{\upsilon-1}\eta(t)\,\mathrm{d}t,$$

in the sense of formal power series; see [7].

Since all initial conditions are zero, we thus obtain the solution

$$f(z) = \left(\mathbf{I}_{\lambda_{d-1}}\circ\cdots\circ\mathbf{I}_{\lambda_1}\circ\mathbf{I}_2\right)[g+2^dB](z), \tag{25}$$

where

$$\mathbf{I}_\upsilon[\phi](z) = (1-z)^{-\upsilon}\int_0^z(1-x)^{\upsilon-1}\phi(x)\,\mathrm{d}x. \tag{26}$$

Note that the function g involves itself f.

Thus the next steps consist of (i) clarifying the changes in asymptotic approximation under consecutive applications of the linear operators, and (ii) simplifying the resulting leading constants.

Asymptotic transfer for the linear operator.

Lemma 1 ([7]). (i) *(Small toll functions) Let $\upsilon\in\mathbb{C}$. If $\int_0^1(1-x)^{\upsilon-1}\phi(x)\,\mathrm{d}x$ converges, then*

$$[z^n]\mathbf{I}_\upsilon[\phi](z) \sim \frac{n^{\upsilon-1}}{\Gamma(\upsilon)}\int_0^1(1-x)^{\upsilon-1}\phi(x)\,\mathrm{d}x, \tag{27}$$

where Γ denotes the Gamma function.

(ii) (Large toll functions) Let $\upsilon \in \mathbb{C}$. If $[z^n]\phi(z) \sim cn^\tau$, where $c \in \mathbb{C}$ and $\Re(\tau) > \Re(\upsilon) - 1$, then

$$[z^n]\, \mathbf{I}_\upsilon[\phi](z) \sim \frac{c}{\tau + 1 - \upsilon} n^\tau. \tag{28}$$

Note that if $\upsilon = 0, -1, \ldots$ in case (i), then the $\sim$-transfer (27) becomes an o-transfer; similarly, if $c = 0$ in case (ii), then (28) becomes an o-transfer.

Proof. (Sketch) The estimate (27) follows from (26), and (28) from the expression

$$[z^n]\, \mathbf{I}_\upsilon[\phi](z) = \frac{\Gamma(n+\upsilon)}{\Gamma(n+1)} \sum_{0 \le k < n} \frac{\Gamma(k+1)}{\Gamma(k+1+\upsilon)} [z^k]\phi(z); \tag{29}$$

see [7]. ∎

Asymptotic linearity. We now prove the small toll functions part of Theorem 2 when $B_n = o(n)$ and $\sum_n B_n n^{-2}$ converges. The assumption that the series $\sum_n B_n n^{-2}$ converges implies that $|\int_0^1 (1 - x)B(x)\,\mathrm{d}x| < \infty$. Assume at the moment that

$$\left| \int_0^1 (1-x) g(x)\,\mathrm{d}x \right| < \infty. \tag{30}$$

Then by applying consecutively Lemma 1, we obtain

$$A_n = [z^n] f(z) + B_n = \frac{K'}{P_0'(2)} n + o(n), \tag{31}$$

where

$$K' := \int_0^1 (1-x) \left(g(x) + 2^d B(x) \right) \mathrm{d}x = \sum_{j \ge 0} \frac{[z^j]g(z) + 2^d B_j}{(j+1)(j+2)}. \tag{32}$$

The next step is to prove (30).

Proof of (30). Define

$$\Lambda(s) := \int_0^1 (1-x)^{s-1} P_0(\vartheta) f(x)\,\mathrm{d}x,$$

where the ϑ-operator is understood to be $(1-x)\mathrm{d}/\mathrm{d}x$.

Since $B_n = o(n) = o(n^{1+\varepsilon})$, $A_n = o(n^{1+\varepsilon})$ by (46) below. Thus $f(x) = O((1-x)^{-2-\varepsilon})$ for $0 \le x < 1$ and

$$P_0(\vartheta) f(x) = O(f^{(d)}(x)) = O((1-x)^{-d-2-\varepsilon}),$$

for $0 \le x < 1$. It follows that $\Lambda(s)$ is finite for sufficiently large s, say $s \ge s_0 \ge d + 2 + \varepsilon$. We show that we can take $s_0 = 2$. Note that $\Lambda(s)$ is an analytic function in the half-plane $\Re(s) \ge 2$, but for our purposes we need only real values of s.

Lemma 2 ([5]). *Let $p(x)$ and $q(x)$ be two polynomials of degrees at most d. Assume that $\phi(x)$ is defined in the unit interval with $\phi^{(j)}(0) = 0$ for $0 \le j < k$. Then*

$$\int_0^1 (1-x)^{s-1} \left(p(\vartheta) q(\vartheta)^{-1} \right) \phi(x)\,\mathrm{d}x = \frac{p(s)}{q(s)} \int_0^1 (1-x)^{s-1} \phi(x)\,\mathrm{d}x, \tag{33}$$

provided that $q(s) \neq 0$ and that both integrals converge.

Substituting (22) into the integral and applying (33), we see that $\Lambda(s)$ satisfies the difference equation

$$\Lambda(s) = 2^d B^*(s) + \sum_{1 \le j < d} \frac{P_j(j+s)}{P_0(j+s)} \Lambda(j+s). \tag{34}$$

By assumption, $B^*(s)$ is finite for $s \ge 2$. Also $\Lambda(s)$ is bounded for $s \ge d+2+\varepsilon$ as showed above. Thus by iterating the equation (34), we deduce that $\Lambda(s)$ is finite for $s \ge 2$.

This proves (30) because

$$\int_0^1 (1-x) g(x) \, \mathrm{d}x = \int_0^1 (1-x) \left(P_0(\vartheta) f(x) - 2^d B(x)\right) \, \mathrm{d}x,$$

and from (32), it follows that $K' = \Lambda(2)$.

Further simplification of the constant K'. Taking first $s = 2$ in (34) and then iterating the recurrence (34) N times, we get

$$K' = K'_N + \sum_{1 \le j \le N(d-1)+1} \frac{e_{N,j}}{P_0(j+N+1)} \Lambda(j+N+1),$$

where $e_{1,j} = P_j(j+2)$ for $1 \le j \le d$,

$$e_{N,j} := \sum_{1 \le \ell \le d} \frac{P_\ell(j+N+1)}{P_0(j+N+1-\ell)} e_{N-1,j+1-\ell} \qquad (1 \le j \le N(d-1)+1),$$

for $N \ge 2$, and

$$K'_N = 2^d \left(B^*(2) + \sum_{1 \le j \le (N-1)d} \frac{B^*(j+2)}{P_0(j+2)} \sum_{1 \le \ell \le j} e_{\ell,j+1-\ell} \right),$$

for $N \ge 0$.

Since $\Lambda(N) \to 0$ as $N \to \infty$, we have

$$K' = \lim_{N \to \infty} K'_N = 2^d \left(B^*(2) + \sum_{j \ge 1} \frac{B^*(j+2)}{P_0(j+2)} \sum_{1 \le \ell \le j} e_{\ell,j+1-\ell} \right).$$

Define

$$V_k := \frac{1}{P_0(k+2)} \sum_{1 \le \ell \le k} e_{\ell,k+1-\ell}.$$

Then V_k satisfies (17) and we have

$$K' = 2^d \sum_{k \ge 0} B^*(k+2) V_k.$$

It follows, by (31), that $K_B = K'/P'_0(2)$.

Absolute convergence of the series representation (16) for K_B. There is no *a priori* reason that the series representation for K_B in (16) is convergent. We show that under the assumptions on B_n in (12) the series in (16) is indeed absolutely convergent.

Observe first that by the factorial series expression in (18)

$$B^*(k+2) = O(k^{-2}).$$

We need then an estimate for V_k.

If $d = 2$, then $P_1(s) = s(s-1)$, and we can solve the recurrence of V_k explicitly, giving

$$V_k = 12\frac{k+1}{(k+3)(k+4)} \qquad (k \geq 0). \tag{35}$$

Consequently,

$$\begin{aligned} K_B &= 12\sum_{k\geq 0}\frac{k+1}{(k+3)(k+4)}\,B^*(k+2) \\ &= 12\int_0^1 B(x)\left(\frac{1+2x}{(1-x)^3}\log\frac{1}{x} - \frac{5+x}{2(1-x)^2}\right)\mathrm{d}x; \end{aligned}$$

see also [36, 39].

Lemma 3. *The sequence V_k satisfies the estimate*

$$V_k = O\left(k^{-1}(\log k)^{d-2}\right), \tag{36}$$

for $d \geq 2$.

The order is tight; indeed, we can derive a more precise asymptotic approximation; see (39) below.

Proof. We first show that the generating function $V(z)$ of V_k satisfies the DE

$$\mathbb{D}_z\left(z(1-z)\mathbb{D}_z\right)^{d-1}\left(z^2V(z)\right) - 2^d zV(z) = 0. \tag{37}$$

By Cauchy's integral representation for V_k

$$V_k = \frac{1}{2\pi i}\oint_{|w|=\varepsilon} w^{-k-1}V(w)\,\mathrm{d}w = \frac{1}{2\pi i}\oint_{|w-1|=\varepsilon}(1-w)^{-k-1}V(1-w)\,\mathrm{d}w.$$

Then, by the relation (see (17)),

$$P_0(k+2)V_k - \sum_{1\leq \ell<d}P_\ell(k+2)V_{k-\ell} = 0,$$

we have

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\oint_{|w-1|=\varepsilon}(1-w)V(1-w)\left[P_0(k+2)(1-w)^{-k-2} - \sum_{1\leq\ell<d}P_\ell(k+2)(1-w)^{-k+\ell-2}\right]\mathrm{d}w \\ &= \frac{1}{2\pi i}\oint_{|w-1|=\varepsilon}(1-w)V(1-w)\left[\vartheta_w(w\vartheta_w)^{d-1} - 2^d\right](1-w)^{-k-2}\,\mathrm{d}w, \end{aligned}$$

by the definition of the P_j's, where $\vartheta_w := (1-w)\mathrm{d}/\mathrm{d}w$. It follows, by multiplying both sides by z^k and then summing over all nonnegative k, that

$$I_d(z) - 2^d V(z) = 0,$$

where

$$I_d(z) := \frac{1}{2\pi i}\oint_{|w-1|=\varepsilon} (1-w)V(1-w)\left[\vartheta_w (w\vartheta_w)^{d-1}\right] \frac{(1-w)^{-2}}{1-\frac{z}{1-w}}\,\mathrm{d}w.$$

By successive integration by parts, we have

$$\begin{aligned} I_d(z) &= \frac{(-1)^d}{2\pi i}\oint_{|w-1|=\varepsilon} \frac{(1-w)^{-2}}{1-\frac{z}{1-w}}\mathbb{D}_w\left(w(1-w)\mathbb{D}_w\right)^{d-1}\left((1-w)^2V(1-w)\right)\,\mathrm{d}w \\ &= \frac{1}{2\pi i}\oint_{|w|=\varepsilon} \frac{w^{-2}}{1-\frac{z}{w}}\mathbb{D}_w\left(w(1-w)\mathbb{D}_w\right)^{d-1}\left(w^2V(w)\right)\,\mathrm{d}w, \end{aligned}$$

where $\mathbb{D}_w := \mathrm{d}/\mathrm{d}w$. This proves (37).

By Frobenius method (see [32]), we seek solutions of the form $V(z) = (1-z)^{-s}\xi(1-z)$ with ξ analytic at zero. Substituting such a form into (37) gives for $d=1$

$$I_1(z) \sim \xi(0)s(1-z)^{-s-1} \qquad (z\sim 1).$$

By induction, we obtain

$$I_d(z) \sim \xi(0)s^d(1-z)^{-s-1} \qquad (z\sim 1).$$

Thus, the indicial equation is $s^d = 0$, implying that

$$V(z) = O\left(\log^{d-1}|1-z|\right) \qquad (z\sim 1).$$

It follows, by singularity analysis (see [21]), that V_k satisfies the estimate (36). This proves Lemma 3. ∎

A more precise approximations to the asymptotics of V_k. Since the generating function of the sequence V_k satisfies the explicit, homogeneous DE (37), we can derive more precise asymptotic estimates as follows.

By applying either the Euler transform approach of [19] or the Poisson generating functions, we obtain

$$V_k = \sum_{1\le \ell\le k+1}\binom{k+1}{\ell}(-1)^{\ell+1}\ell\prod_{1\le j<d}\frac{\Gamma(3-\lambda_j)\Gamma(\ell+1)}{\Gamma(\ell+2-\lambda_j)} \qquad (k\ge 0).$$

Consequently, we have the integral representation (see [23])

$$V_k = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{\Gamma(k+2)\Gamma(1-s)}{\Gamma(k+2-s)}\prod_{1\le j<d}\frac{\Gamma(3-\lambda_j)\Gamma(s+1)}{\Gamma(s+2-\lambda_j)}\,\mathrm{d}s. \tag{38}$$

From this representation, we can show that

$$V_k \sim \frac{d2^{d-1}(2^d-1)}{(d-2)!}k^{-1}(\log k)^{d-2}, \tag{39}$$

for $d \geq 2$ and large k. Note that the leading constants first grows and then decreases to zero

$$\left\{\frac{d2^{d-1}(2^d-1)}{(d-2)!}\right\}_{d\geq 2} = \left\{12, 84, 240, 413\tfrac{1}{3}, 504, 474\tfrac{2}{15}, 362\tfrac{2}{3}, 233\tfrac{3}{5}, 129\tfrac{19}{21}, 63\tfrac{1531}{2835}, \cdots\right\}.$$

Since the leading constants are quite large for small d, the convergence of the series (16) is poor for small d; we will propose a more efficient numeric procedure for computing K_B.

In particular, if $d = 2$, the integrand has three simple poles at $s = -1, -2$, and -3, and the residues of these poles add up to $12(k+1)/((k+3)(k+4))$, in accordance with (35). But for $d \geq 3$, the resulting expressions are more complicated because there are infinitely many poles.

An integral representation for the constant K_B. By substituting the expression (38) of V_k in (16), we obtain

$$K_B = \frac{2}{2d\pi i}\int_{c-i\infty}^{c+i\infty} \Upsilon(s) \prod_{1\leq j<d} \frac{\Gamma(3-\lambda_j)\Gamma(s+1)}{\Gamma(s+2-\lambda_j)}\,\mathrm{d}s, \tag{40}$$

where

$$\Upsilon(s) := \sum_{k\geq 0} B^*(k+2)\frac{\Gamma(k+2)\Gamma(1-s)}{\Gamma(k+2-s)},$$

and $c > -1$ lies in the half-plane where the series on the right-hand side converges. Thus if analytic properties of Υ are known, then K_B can be further simplified; see for example (44). Also if $d = 2$, then $K_B = 12(\Upsilon(-1) - 2\Upsilon(-2) + \Upsilon(-3))$; see (35).

Nonzero initial conditions. We now prove that the linearity constant K_B is of the form (16) even with nonzero initial conditions.

We start from making all the initial conditions zero

$$\bar{f}(z) := f(z) - \sum_{0\leq j<d} (A_j - B_j)z^j,$$

so that, by (21),

$$\left(\vartheta(z\vartheta)^{d-1} - 2^d\right)\bar{f}(z) = 2^d B(z) + 2^d C(z),$$

where (for convenience, defining $B_0 = 0$)

$$C(z) := \sum_{0\leq j<d} (A_j - B_j)\,z^j - 2^{-d}\left(\vartheta(z\vartheta)^{d-1}\right)\left(\sum_{0\leq j<d}(A_j - B_j)z^j\right).$$

By the same approach as above, we obtain $A_n \sim \bar{K}n$, where the linearity constant $\bar{K}$ is given by

$$\bar{K} = \frac{2}{d}\sum_{k\geq 0} V_k B^*(k+2) + \frac{2}{d}\sum_{k\geq 0} V_k \int_0^1 (1-x)^{k+1} \sum_{0\leq j<d}(A_j - B_j)x^j\,\mathrm{d}x + \bar{c}.$$

Here

$$\begin{aligned}
\bar{c} &:= -\frac{2^{1-d}}{d}\sum_{k\geq 0} V_k \int_0^1 (1-x)^{k+1}\left(\vartheta_x(x\vartheta_x)^{d-1}\right)\left(\sum_{0\leq j<d}(A_j - B_j)x^j\right)\mathrm{d}x \\
&= -\frac{2^{1-d}}{d}\int_0^1 (1-x)V(1-x)\left(\vartheta_x(x\vartheta_x)^{d-1}\right)\left(\sum_{0\leq j<d}(A_j - B_j)x^j\right)\mathrm{d}x.
\end{aligned}$$

By the same argument used to derive the DE satisfied by $V(z)$, we have

$$\bar{c} = -\frac{2^{1-d}}{d}\int_0^1 \left(\sum_{0\le j<d}(A_j-B_j)(1-x)^j\right)\mathbb{D}_x\left(x(1-x)\mathbb{D}_x\right)^{d-1}\left(x^2V(x)\right)\,\mathrm{d}x.$$

But by (37)

$$\mathbb{D}_x\left(x(1-x)\mathbb{D}_x\right)^{d-1}\left(x^2V(x)\right) = 2^d xV(x);$$

it follows that

$$\bar{c} = -\frac{2}{d}\int_0^1 (1-x)V(1-x)\left(\sum_{0\le j<d}(A_j-B_j)x^j\right)\,\mathrm{d}x.$$

Thus

$$\bar{K} = \frac{2}{d}\sum_{k\ge 0} V_k B^*(k+2);$$

this proves that the linearity constant is of the same form (16), which amounts to saying that *we do not need to nullify the initial conditions.*

An efficient numeric procedure. The above proof suggests a useful numeric procedure for computing the constant K_B. The crucial observation is that the first d terms we choose to be subtracted from $\bar{f}$ play no special role in our proof, meaning that we can indeed subtract a sufficiently large number, say N, of initial terms from f, resulting in a series form for K_B with convergence rate $(\log k)^{d-2}k^{-N}$. This is because the right-hand side of the DE is of order z^{N-1}, which yields, after taking the finite Mellin transform, the order k^{-N} for large k. Such a procedure quickly leads to a good numeric approximation to the leading constant K_B to high precision. We will apply this procedure to the constants appearing in the mean and variance of the number of leaves in Section 3.2

Necessity in (12). Assume that $A_n \sim cn$ for some constant c. The special form (8) or the following one (see [19])

$$\pi_{n,j} = \frac{1}{(d-1)!}\binom{n-1}{j}\int_0^1 (-\log t)^{d-1}t^j(1-t)^{n-1-j}\,\mathrm{d}t,$$

can be used to prove that $B_n = o(n)$ by (7). We propose instead a proof based again on linear operators, the advantage being generally applicable to more complicated recurrences while keeping uniformity of the proof.

By (21)

$$\begin{aligned}B(z) &= A(z) - 2^d\left(\vartheta^{-1}\left(z^{-1}\vartheta^{-1}\right)^{d-1}\right)A(z)\\ &= A(z) - 2^d\left(\mathbf{I}_0\circ\left(z^{-1}\mathbf{I}_0\right)^{d-1}\right)[A](z).\end{aligned}$$

Since $A_n \sim cn$, we have, by (28),

$$[z^n]\mathbf{I}_0[A](z) \sim \frac{c}{2}n, \quad [z^n]z^{-1}\mathbf{I}_0[A](z) \sim \frac{c}{2}n.$$

Applying successively these estimates yields

$$[z^n]2^d\left(\mathbf{I}_0\circ\left(z^{-1}\mathbf{I}_0\right)^{d-1}\right)[A](z) \sim cn.$$

Thus $B_n = o(n)$.

We then prove that $|\sum_n B_n n^{-2}| < \infty$ by showing that $B^*(2)$ is finite. By (34), it suffices to show that $\Lambda(2)$ is finite. Since $A_n \sim cn$ and $B_n = o(n)$, we deduce that $f(x) = O((1-x)^{-2})$ for $0 \le x < 1$. It follows that

$$\begin{aligned}\Lambda(2) &= \lim_{s\to 2^+} \Lambda(s)\\ &= \lim_{s\to 2^+} P_0(s)\int_0^1 (1-x)^{s-1} f(x)\,\mathrm{d}x\\ &= O(1).\end{aligned}$$

This complete the proof of (12).

2.3 Asymptotic transfer II. Linear toll functions

We prove part (ii) of Theorem 2 in this section. By the result of part (i), it suffices to consider the case when $B_n \equiv n$ for $n \ge 1$. Then $B(z) = z/(1-z)^2$.

All initial conditions zero. It is simpler, as in part (i), to consider

$$\bar{f}(z) := A(z) - B(z) - \sum_{0\le j<d} (A_j - B_j) z^j,$$

so that $\bar{f}$ satisfies the DE

$$\left(\vartheta(z\vartheta)^{d-1} - 2^d\right)\bar{f}(z) = 2^d B(z) + 2^d C(z),$$

with zero initial conditions, where

$$C(z) := \left(2^{-d}\vartheta(z\vartheta)^{d-1} - 1\right)\sum_{1\le j<d}(A_j - B_j)z^j.$$

Then $\bar{f}$ satisfies the DE

$$P_0(\vartheta)\bar{f}(z) = 2^d B(z) + 2^d C(z) + g(z),$$

where g is defined in (22), and for $n \ge d$

$$\begin{aligned}A_n &= [z^n]\left(\bar{f}(z) + B(z)\right)\\ &= n + [z^n]\left(\mathbf{I}_{\lambda_{d-1}}\cdots\circ\mathbf{I}_{\lambda_1}\circ\mathbf{I}_2\right)\left[2^d B + 2^d C + g\right](z).\end{aligned}$$

An expression for the iterates of the I-operators. Observe first that by integration by parts

$$\left(\mathbf{I}_\upsilon\circ\mathbf{I}_\tau\right)[\xi](z) = \frac{1}{\tau-\upsilon}\mathbf{I}_\tau[\xi](z) - \frac{1}{\tau-\upsilon}\mathbf{I}_\upsilon[\xi](z) \qquad (\upsilon\ne\tau),$$

so that by induction

$$\left(\mathbf{I}_{\lambda_{d-1}}\circ\cdots\circ\mathbf{I}_{\lambda_0}\right)[\xi](z) = \sum_{0\le j<d}\frac{\mathbf{I}_{\lambda_j}[\xi](z)}{\prod_{\ell\ne j}(\lambda_j-\lambda_\ell)}. \tag{41}$$

Thus

$$\bar{f}(z) = \sum_{0\le j<d}\frac{\mathbf{I}_{\lambda_j}[2^d B + 2^d C + g](z)}{P_0'(\lambda_j)}.$$

The contribution of $2^d B(z)$. By applying (41), we have

$$\begin{aligned}[z^n]\left(\mathbf{I}_{\lambda_{d-1}}\circ\cdots\circ\mathbf{I}_{\lambda_1}\circ\mathbf{I}_2\right)[2^dB](z) &= \sum_{0\le j<d}\frac{2^d}{P_0'(\lambda_j)}[z^n]\mathbf{I}_{\lambda_j}[B](z)\\ &= \frac{2^d}{P_0'(2)}[z^n]\left((1-z)^{-2}\log\frac{1}{1-z}-(1-z)^{-2}\right)\\ &\quad+\sum_{1\le j<d}\frac{2^d}{(2-\lambda_j)P_0'(\lambda_j)}[z^n](1-z)^{-2}+o(n).\end{aligned}$$

Now

$$\begin{aligned}\sum_{1\le j<d}\frac{2^d}{(2-\lambda_j)P_0'(\lambda_j)} &= \frac{1}{d}\sum_{1\le j<d}\frac{\lambda_j}{2-\lambda_j}\\ &= \frac{2}{d}\sum_{1\le j<d}\frac{1}{2-\lambda_j}-\frac{d-1}{d}\\ &= \frac{P_0''(2)}{dP_0'(2)}-\frac{d-1}{d}\\ &= -\frac{d-1}{2d}.\end{aligned}$$

Thus

$$\begin{aligned}&[z^n]\left(\mathbf{I}_{\lambda_{d-1}}\circ\cdots\circ\mathbf{I}_{\lambda_1}\circ\mathbf{I}_2\right)[2^dB](z)\\ &\quad= [z^n]\left(\frac{2}{d}\frac{1}{(1-z)^2}\log\frac{1}{1-z}-\frac{d+3}{2d}\frac{1}{(1-z)^2}\right)+o(n)\\ &\quad= \frac{2}{d}n\log n+\left(\frac{2\gamma}{d}-\frac{1}{2}-\frac{7}{2d}\right)n+o(n),\end{aligned}\tag{42}$$

since

$$\begin{aligned}[z^n](1-z)^{-2}\log\frac{1}{1-z} &= (n+1)\sum_{1\le j\le n}j^{-1}-n\\ &= n\log n+(\gamma-1)n+O(\log n).\end{aligned}$$

The contribution of $2^dC(z)$ and $g(z)$. Similarly, by (27),

$$[z^n]\left(\mathbf{I}_{\lambda_{d-1}}\circ\cdots\circ\mathbf{I}_{\lambda_1}\circ\mathbf{I}_2\right)[2^dC](z) = \frac{2}{d}C^*(2)n+o(n),$$

where $C^*(s) := \int_0^1 C(x)(1-x)^{s-1}\,\mathrm{d}x$, and

$$\begin{aligned}[z^n]\left(\mathbf{I}_{\lambda_{d-1}}\circ\cdots\circ\mathbf{I}_{\lambda_1}\circ\mathbf{I}_2\right)[g](z) &= \frac{2^{1-d}}{d}[z^n]\mathbf{I}_2[g](z)+o(n)\\ &= \frac{2^{1-d}}{d}g^*(2)n+o(n),\end{aligned}$$

provided that $g^*(2)$ is finite, where $g^*(s) := \int_0^1(1-x)^{s-1}g(x)\,\mathrm{d}x$.

Boundness of $g^*(2)$. To justify that $g^*(2)$ is finite, we use the same argument as in the proof for $\Lambda(s)$ above. Again by Lemma 2

$$
\begin{aligned}
g^*(s) &= \sum_{1\le j<d} \int_0^1 (1-x)^{j+s-1} P_j(\vartheta) P_0(\vartheta)^{-1} \left(2^d B(x) + 2^d C(x) + g(x)\right) \,\mathrm{d}x \\
&= \sum_{1\le j<d} \frac{P_j(j+s)}{P_0(j+s)} \int_0^1 (1-x)^{j+s-1} \left(2^d B(x) + 2^d C(x) + g(x)\right) \,\mathrm{d}x \\
&= \sum_{1\le j<d} \frac{P_j(j+s)}{P_0(j+s)} \left(2^d B^*(j+s) + 2^d C^*(j+s) + g^*(j+s)\right),
\end{aligned}
$$

where

$$
B^*(s) = \int_0^1 x(1-x)^{s-3}\,\mathrm{d}x = \frac{1}{(s-1)(s-2)}.
$$

Since $B^*(s)$ is finite for $s > 2$, $g^*(s)$ is well-defined for $s > 1$.

Iterating the recurrence as in part (i) gives

$$
\begin{aligned}
g^*(2) &= \sum_{j\ge 0} V_j \sum_{1\le \ell<d} \frac{P_\ell(j+\ell+2)}{P_0(j+\ell+2)} \left(2^d B^*(j+\ell+2) + 2^d C^*(j+\ell+2)\right) \\
&= \sum_{k\ge 1} \left(2^d B^*(k+2) + 2^d C^*(k+2)\right) \sum_{1\le \ell<d} \frac{P_\ell(k+2)}{P_0(k+2)} V_{k-\ell} \\
&= 2^d \sum_{k\ge 1} \frac{V_k}{k(k+1)} + 2^d \sum_{k\ge 1} V_k C^*(k+2),
\end{aligned}
$$

where V_k is defined in (17).

Collecting all estimates. Combining this with (42), we obtain

$$
A_n = \frac{2}{d} n \log n + K_2 n + o(n),
$$

where

$$
K_2 = \frac{2\gamma}{d} + \frac{1}{2} - \frac{7}{2d} + \frac{2}{d} \sum_{k\ge 1} \frac{V_k}{k(k+1)} + \frac{2}{d} \sum_{k\ge 0} V_k C^*(k+2).
$$

The last series $\sum_{k\ge 0} V_k C^*(k+2)$ is identically zero by the same argument used in part (i) for nonzero initial conditions.

Final simplification. We now show that

$$
\sum_{k\ge 1} \frac{V_k}{k(k+1)} = \sum_{1\le j<d} \psi(3-\lambda_j) - (d-1)(1-\gamma), \tag{43}
$$

and this will prove (14) by the relations $\psi(3-\lambda_j) = \psi(2-\lambda_j) + (2-\lambda_j)^{-1}$ and

$$
\sum_{1\le j<d} \frac{1}{2-\lambda_j} = \frac{d-1}{4}.
$$

For that purpose, we substitute the integral representation (38) into the series and then sum over all positive indices k, giving

$$\sum_{k\geq 1}\frac{V_k}{k(k+1)}=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{1}{(s-1)^2}\prod_{1\leq j<d}\frac{\Gamma(3-\lambda_j)\Gamma(s+1)}{\Gamma(s+2-\lambda_j)}\,\mathrm{d}s. \tag{44}$$

Moving the line of integration to the right and taking into account the residue of the unique pole encountered at $s=1$, we obtain (43) by absolute convergence.

A different expression for K_2. Yet another expression for K_2 was derived in [19]

$$K_2=\frac{2\gamma}{d}+\frac{3}{2}-\frac{3}{2d}-2^{d+1}\sum_{k\geq 3}\frac{1}{k(k^d-2^d)}.$$

Equating the two expressions of K_2 leads to the identity

$$2^{d+1}\sum_{k\geq 3}\frac{1}{k(k^d-2^d)}=3-\frac{2}{d}(d-1)\gamma-\frac{2}{d}\sum_{1\leq j<d}\psi(3-\lambda_j)\qquad (d\geq 1),$$

which can be proved using the relations

$$\psi(z+1)=-\gamma+\sum_{j\geq 1}\frac{z}{j(j+z)},$$

(see [14, p.15, Eq. (3)]) and

$$\begin{aligned}\sum_{1\leq j<d}\frac{2-\lambda_j}{k+2-\lambda_j}&=d-1-k\sum_{1\leq j<d}\frac{1}{k+2-\lambda_j}\\&=d-1-k\left(\frac{d(k+2)^{d-1}}{(k+2)^d-2^d}-\frac{1}{k}\right).\end{aligned}$$

Necessity. Consider the case when $A_n=c_0 n\log n+c_1 n+o(n)$, where $c_0=2/d$. Then, similarly as in part (i), we need the elementary estimate

$$\begin{aligned}[z^n]\mathbf{I}_0[A](z)&=\frac{1}{n}\sum_{0\leq j<n}A_j\\&=\frac{1}{n}\sum_{1\leq j<n}(c_0 j\log j+c_1 j)+o(n)\\&=\frac{c_0}{2}n\log n+\left(\frac{c_1}{2}-\frac{c_0}{4}\right)n+o(n).\end{aligned}$$

The same estimate holds for $[z^n]z^{-1}\mathbf{I}_0[A](z)$. Iterating the estimates, we obtain

$$[z^n]2^d\left(\mathbf{I}_0\circ\left(z^{-1}\mathbf{I}_0\right)^{d-1}\right)[A](z)=c_0 n\log n+\left(c_1-\frac{d}{2}c_0\right)n+o(n).$$

Consequently,

$$B_n=\frac{d}{2}c_0 n+o(n)=n+o(n).$$

Thus $B_n-n=o(n)$ and the remaining proof uses the same argument as in part (i). This completes the proof of (13).

2.4 Asymptotic transfer III. Large toll functions

We prove the asymptotic transfer (15) for large toll functions. For general divide-and-conquer recurrences, such a case is always easier than that of small toll functions, one simple reason being that the major contribution comes from a few large terms instead of summing over all small parts like the small toll functions case. More precisely, we expect that most contribution comes from the term $2^d B(z)$ in (22), the other term $g(z)$ being asymptotically negligible.

Assume that $B_n \sim cn^\upsilon$, where $\upsilon > 1$. We start again from (25), which gives

$$\begin{aligned} A_n &= B_n + [z^n] \left(\mathbf{I}_{\lambda_{d-1}} \circ \cdots \mathbf{I}_{\lambda_1} \circ \mathbf{I}_2\right) [g + 2^d B](z) \\ &= B_n + A_n^{[1]} + A_n^{[2]}, \end{aligned}$$

where, by successive applications of (28), we have

$$\begin{aligned} A_n^{[2]} &:= 2^d [z^n] \left(\mathbf{I}_{\lambda_{d-1}} \circ \cdots \mathbf{I}_{\lambda_1} \circ \mathbf{I}_2\right) [B](z) \\ &\sim \frac{c2^d}{P_0(\upsilon+1)} n^\upsilon. \end{aligned}$$

To estimate $A_n^{[1]}$, we first consider $g^*(s) = \int_0^1 (1-x)^{s-1} g(x)\,\mathrm{d}x$, which, by (34), satisfies the recurrence equation

$$\begin{aligned} g^*(s) &= \int_0^1 (1-x)^{s-1} \left(P_0(\vartheta) f(x) - 2^d B(x)\right)\,\mathrm{d}x \\ &= \sum_{1\le j<d} \frac{P_j(j+s)}{P_0(j+s)} \left(g^*(j+s) + 2^d B^*(j+s)\right), \end{aligned} \tag{45}$$

for sufficiently large s. Since $B_n \sim cn^\upsilon$, we deduce that $B^*(s)$ is finite for $s > \upsilon + 1$. The same argument as for $\Lambda(s)$ shows that $g^*(s)$ is finite for $s > \upsilon$. This implies, in particular, that

$$\left|\int_0^1 (1-x)^{\upsilon-\varepsilon} g(x)\,\mathrm{d}x\right| = \left|\Gamma(\upsilon+1-\varepsilon) \sum_{k\ge 0} \frac{\Gamma(k+1)}{\Gamma(k+\upsilon+2-\varepsilon)} [z^k]g(z)\right| < \infty.$$

Now by (29) with $\upsilon = 2$

$$[z^n]\mathbf{I}_2[g](z) = (n+1) \sum_{0\le k<n} \frac{[z^k]g(z)}{(k+1)(k+2)}.$$

Let $S_k := \sum_{0\le j\le k} \Gamma(j+1)[z^j]g(z)/\Gamma(j+\upsilon+2-\varepsilon)$. Then $S_k = O(1)$ and, by partial summation,

$$\begin{aligned} (n+1) \sum_{0\le k<n} \frac{[z^k]g(z)}{(k+1)(k+2)} &= (n+1) \sum_{0\le k<n} \frac{\Gamma(k+1)}{\Gamma(k+\upsilon+2-\varepsilon)} [z^k]g(z) \cdot \frac{\Gamma(k+\upsilon+2-\varepsilon)}{\Gamma(k+3)} \\ &= (1-\upsilon+\varepsilon)(n+1) \sum_{0\le k\le n} S_k \frac{\Gamma(k+\upsilon+2-\varepsilon)}{\Gamma(k+4)} + O(n^{\upsilon-\varepsilon}) \\ &= O(n^{\upsilon-\varepsilon}). \end{aligned}$$

Applying now successively (28), we obtain $A_n^{[1]} = O(n^{\upsilon-\varepsilon}) = o(n^\upsilon)$.

From these estimates, it follows that

$$A_n \sim cn^\upsilon + \frac{c2^d}{(\upsilon+1)^d - 2^d} n^\upsilon,$$

which implies the sufficiency part of (15).

Necessity in (15). Assume that $A_n \sim K_3 c n^\upsilon$, where $K_3 = (\upsilon+1)^d/((\upsilon+1)^d - 2^d)$. Then, similarly to the necessity proof for case (i),

$$[z^n]2^d \left(\mathbf{I}_0 \circ \left(z^{-1}\mathbf{I}_0\right)^{d-1}\right)[A](z) \sim \frac{2^d}{(\upsilon+1)^d} n^\upsilon,$$

by successive applications of (28). Then

$$B_n \sim K_3 c \left(1 - \frac{2^d}{(\upsilon+1)^d}\right) n^\upsilon \sim c n^\upsilon.$$

Simple transfers for the quadtree recurrence (7). The same proof also gives the following O- and o-transfers.

Lemma 4. *Assume $\upsilon > 1$. Then*

$$B_n = O(n^\upsilon) \quad \textit{iff} \quad A_n = O(n^\upsilon). \tag{46}$$

The same result holds with O replaced by o.

Note that the results for large toll functions can also be proved by other elementary means, but the proof given here based on iterative operators applies for all cases, and is thus more general and uniform.

Recurrence of the Cauchy-Euler part. The preceding analysis shows that when B_n is larger than linear, the contribution from $g(z)$ to A_n is asymptotically negligible. Thus in this case $A_n \sim A_n^{[2]}$, where $P_0(\vartheta)(A^{[2]}(z) - B(z)) = 2^d B(z)$, or in terms of recurrence

$$A_n^{[2]} = B_n + 2^d \sum_{0\le j<n} \tilde{\pi}_{n,j} A_j^{[2]},$$

where

$$\tilde{\pi}_{n,j} = \frac{1}{n} \sum_{j<j_1<\cdots<j_{d-1}<n} \frac{1}{j_1 \cdots j_{d-1}},$$

which is to be compared with the alternative expression for $\pi_{n,j}$ (see [19])

$$\pi_{n,j} = \frac{1}{n} \sum_{j<j_1\le\cdots\le j_{d-1}\le n} \frac{1}{j_1 \cdots j_{d-1}}.$$

2.5 Asymptotic transfer IV. Further refinements

When more precise information on B_n is available, we can refine the preceding approach and obtain more effective approximations to A_n. We consider the following two cases for later use. Recall that $2e^{2\pi i/d} = \alpha + 1 + i\beta$.

Proposition 1. *Assume that A_n satisfies (7).*

(i) *If $B_n \sim c n^\upsilon$, where $c, \upsilon \in \mathbb{C}$ and $\alpha < \Re(\upsilon) < 1$, then*

$$A_n = K_B n + \frac{c(\upsilon+1)^d}{(\upsilon+1)^d - 2^d} n^\upsilon + o(n^{\Re(\upsilon)} + n^\varepsilon),$$

where K_B is defined in (16).

(ii) *If $B_n = o(n^\alpha)$, then*

$$A_n = K_B n + K(\lambda_1) n^{\alpha+i\beta} + K(\lambda_2) n^{\alpha-i\beta} + o(n^\alpha + n^\varepsilon), \tag{47}$$

where the $K(\lambda_j)$'s are defined in (48). If the B_k's are all real, then $K(\lambda_1) = \overline{K(\lambda_2)}$.

Proof. The proof consists of refining the analysis for the small toll functions part of Theorem 2 using the arguments for large toll functions.

Case (i)**.** Since $B_n \sim cn^\upsilon$, the series in (12) obviously converges. Thus, by (29), we first have

$$\begin{aligned}
[z^n]\mathbf{I}_2[g + 2^d B](z) &= (n+1) \left(\sum_{k \ge 0} \frac{g_k + 2^d B_k}{(k+1)(k+2)} - \sum_{k \ge n} \frac{g_k + 2^d B_k}{(k+1)(k+2)} \right) \\
&= K' n - n \sum_{k \ge n} \frac{g_k}{(k+1)(k+2)} - \frac{c 2^d}{1-\upsilon} n^\upsilon + o(n^{\Re(\upsilon)}) + O(1),
\end{aligned}$$

where $g_k := [z^k]g(z)$ and $K' = \int_0^1 (1-x)\left(g(x) + 2^d B(x)\right) \mathrm{d}x$.

By the same arguments used for $g^*(s)$ in (45), we deduce that $B^*(s)$ is finite for $s > \Re(\upsilon) + 1$ and $g^*(s)$ is bounded for $s > \Re(\upsilon)$. It follows, by the same summation by parts argument used for $A_n^{[1]}$, that

$$n \sum_{k \ge n} \frac{g_k}{(k+1)(k+2)} = O\left(n^{\Re(\upsilon) - \varepsilon}\right).$$

Thus

$$[z^n]\mathbf{I}_2[g + 2^d B](z) = K' n - \frac{c 2^d}{1-\upsilon} n^\upsilon + o(n^{\Re(\upsilon)}) + O(1).$$

We may assume that $\Re(\upsilon) > 0$; otherwise all error terms are absorbed in $o(n^\varepsilon)$.
Consider now

$$\begin{aligned}
[z^n]\left(\mathbf{I}_{\lambda_1} \circ \mathbf{I}_2\right)[g + 2^d B](z) &= \frac{\Gamma(n+\lambda_1)}{\Gamma(n+1)} \sum_{0 \le k < n} \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda_1)} \left(K' k - \frac{c 2^d}{1-\upsilon} k^\upsilon + o(k^{\Re(\upsilon)} + k^\varepsilon) \right) \\
&= \frac{K'}{2-\lambda_1} n - \frac{c 2^d}{(1-\upsilon)(\upsilon+1-\lambda_1)} n^\upsilon + o(n^{\Re(\upsilon)} + n^\varepsilon),
\end{aligned}$$

again by (29). Repeating the same procedure, we obtain

$$A_n - B_n = [z^n] f(z) = \frac{K'}{P_0'(2)} n + \frac{c\, 2^d}{P_0(\upsilon+1)} n^\upsilon + o(n^{\Re(\upsilon)} + n^\varepsilon),$$

which proves (i) since $K_B = K'/P_0'(2)$.

Case (ii)**.** Now, similarly as above, we have

$$\begin{aligned}
[z^n]\mathbf{I}_2[g + 2^d B](z) &= K' n + o(n^\alpha + n^\varepsilon), \\
[z^n]\mathbf{I}_{\lambda_j}[g + 2^d B](z) &= K_j' n^{\lambda_j - 1} + o(n^\alpha + n^\varepsilon),
\end{aligned}$$

where

$$K_j' := \frac{1}{\Gamma(\lambda_j)} \int_0^1 (1-x)^{\lambda_j - 1} \left(g(x) + 2^d B(x)\right) \, \mathrm{d}x \qquad (j = 1, 2).$$

Substituting these estimates into (41) gives

$$[z^n] \left(\mathbf{I}_{\lambda_2} \circ \mathbf{I}_{\lambda_1} \circ \mathbf{I}_2\right) [g + 2^d B](z) = \frac{K'}{(2-\lambda_1)(2-\lambda_2)} n + \frac{K_1'}{(\lambda_1 - 2)(\lambda_1 - \lambda_2)} n^{\lambda_1 - 1}$$
$$+ \frac{K_2'}{(\lambda_2 - 2)(\lambda_2 - \lambda_1)} n^{\lambda_2 - 1} + o(n^\alpha + n^\varepsilon).$$

Applying successively (28) to the remaining operators $\mathbf{I}_{\lambda_j}$ for $j = 3, \dots d-1$, we obtain (47), where

$$K(\lambda_j) = \frac{2^d}{P_0'(\lambda_j)\Gamma(\lambda_j)} \sum_{k \ge 0} B^*(\lambda_j + k) V_k(\lambda_j) \qquad (j = 1, 2), \tag{48}$$

where $V_k(\lambda_j)$ satisfies the recurrence

$$V_k(\lambda_j) = \sum_{1 \le \ell < d} \frac{P_i(\lambda_j + \ell)}{P_0(\lambda_j + \ell)} V_{k-\ell}(\lambda_j),$$

with $V_k(\lambda_j) = 0$ if $k < 0$ and $V_0(\lambda_j) = 1$.

The same proof for proving Lemma 3 also implies that $V_k(\lambda_j)$ satisfies the DE

$$\mathbb{D}_z \left(z(1-z)\mathbb{D}_z\right)^{d-1} \left(z^{\lambda_j} V(z)\right) - 2^d z^{\lambda_j - 1} V(z) = 0,$$

and it follows that $V_k(\lambda_j) = O\left(k^{-1} (\log k)^{d-2}\right)$. This justifies the absolute convergence of the series (48). ∎

In a similar way, we also have the following simpler transfer.

Corollary 1. *Assume that $\Re(\upsilon) < 1$ and $\upsilon \ne \alpha \pm i\beta$. If $B_n = O(n^{\Re(\upsilon)})$, then $A_n = K_B n + O(n^{\Re(\upsilon)} + n^\alpha + n^\varepsilon)$; if $B_n = o(n^{\Re(\upsilon)})$, then $A_n = K_B n + o(n^{\Re(\upsilon)}) + O(n^\alpha + n^\varepsilon)$.*

3 Limit laws of X_n: from normal to periodic

We prove first Theorem 1 in this section. Although the first part of Theorem 1 is implied by Theorem 4 below, we give the main steps of the proof by the moment-transfer approach for more logical reasons: first the mean and variance are needed by both proofs (although with different degrees of precision); second, the main hard part of the proof of Theorem 4 consists in refining the estimates of some recursive functionals of moments. We then sketch extensions of the same types of limit results to other toll functions.

The proofs here rely strongly on the different types of asymptotic transfer we developed in Section 2.

3.1 Limit theorems for the number of leaves

Expected number of leaves. By (5), we see that the mean number of leaves in a random quadtree of n nodes satisfies the recurrence (7) with $B_n = \delta_{n,1}$ and $A_0 = 0$. Then $B(x) = x$ and $B^*(s) = s^{-1}(s+1)^{-1}$. Applying (47), we obtain

$$\mathbb{E}(X_n) = \mu_d n + c_+ n^{\alpha + i\beta} + c_- n^{\alpha - i\beta} + o(n^\alpha + n^\varepsilon), \tag{49}$$

for $d \geq 1$, where $c_+ = K(\lambda_1)$ and $c_- = K(\lambda_2)$ with $B^*(s) = s^{-1}(s+1)^{-1}$. In particular,

$$\mu_d = \frac{2}{d} \sum_{k \geq 0} \frac{V_k}{(k+2)(k+3)}.$$

This proves (3) with $G_1(x) = c_+ e^{i\beta x} + c_- e^{-i\beta x}$; see Figure 2 for a plot of the fluctuations of the error terms. We now show that

$$\mu_d = \frac{2^{d+1}}{d} \sum_{k \geq 2} \frac{1}{k^d [k]!} \left((k-1) \sum_{1 \leq j < d} (\psi(k+1-\lambda_j) - \psi(k)) - 2 \right), \tag{50}$$

for $d \geq 2$, which gives an alternative expression to (2).

To prove (50), we apply the integral representation (40), where

$$\begin{aligned} \Upsilon(s) &:= \sum_{k \geq 0} \frac{\Gamma(k+2)\Gamma(1-s)}{(k+2)(k+3)\Gamma(k+2-s)} \\ &= s^2 \psi'(-s) + s - \frac{1}{2} \qquad (\Re(s) < 1). \end{aligned}$$

Now Υ has double poles at all positive integers. Summing over all residues of the double poles of the integrand in (40), we obtain (50) by absolute convergence (since $\Upsilon(s) = O(|s|^{-1})$ as $|s| \to \infty$ and s is at least ε away from all positive integers). Note that

$$(k-1) \sum_{1 \leq j < d} (\psi(k+1-\lambda_j) - \psi(k)) - 2 = d - 1 + O(k^{-1});$$

thus the general terms in (50) decrease at the rate $O(k^{-d})$.

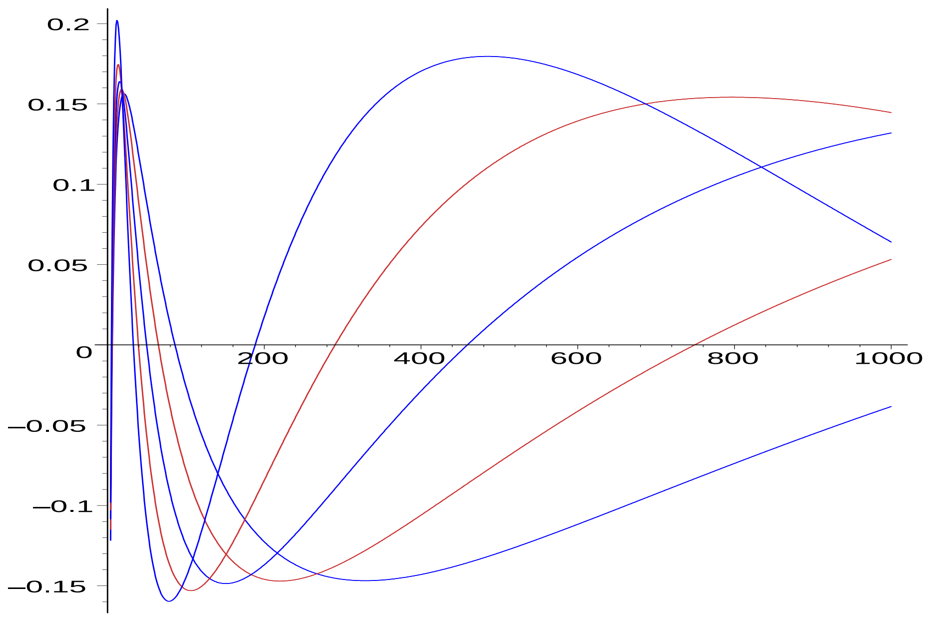

Figure 2: *Periodic fluctuations of* $n^{-\alpha}(\mathbb{E}(X_n) - \mu_d n)$ *for* $n = 4, \ldots, 1000$ *and* $d = 6, \ldots, 10$.

Recurrence of higher moments. For higher moments, we start from the by now standard trick of shifting the mean; thus we consider the moment generating function

$$M_n(y) := \mathbb{E}\left(\exp\left(X_n - \mu_d n - \frac{\mu_d}{2^d-1}\right)y\right),$$

which satisfies, by (5), the recurrence

$$M_n(y) = \sum_{j_1+\cdots+j_{2^d}=n-1} \pi_{n,\mathbf{j}} M_{j_1}(y)\cdots M_{j_{2^d}}(y) \qquad (n\ge 2),$$

with the initial conditions $M_0(y) = e^{-\mu_d y/(2^d-1)}$ and $M_1(y) = e^{(1-2^d\mu_d/(2^d-1))y}$. Note that the additional factor $\mu_d/(2^d-1)$ subtracted has the effect of keeping the recurrence simpler.

Define $M_{n,k} := M_n^{(k)}(0) = \mathbb{E}\left((X_n - \mu_d n - \mu_d/(2^d-1))^k\right)$. Then $M_{n,k}$ satisfies the recurrence

$$M_{n,k} = Q_{n,k} + 2^d \sum_{0\le j<n} \pi_{n,j} M_{j,k} \qquad (n\ge 2),$$

with the initial conditions $M_{0,k} = (-1)^k\mu_d^k/(2^d-1)^k$ and $M_{1,k} = (1-2^d\mu_d/(2^d-1))^k$, where

$$Q_{n,k} = \sum_{\substack{j_1+\cdots+j_{2^d}=n-1\\ i_1+\cdots+i_{2^d}=k\\ i_1,\dots,i_{2^d}<k}} \binom{k}{i_1,\dots,i_{2^d}} \pi_{n,\mathbf{j}} M_{j_1,i_1}\cdots M_{j_{2^d},i_{2^d}} \qquad (n\ge 2).$$

Note that by (3)

$$M_{n,1} = \begin{cases} O\left(n^\alpha + n^\varepsilon\right), & \text{if } 1\le d\le 8;\\ G_1(\beta\log n)n^\alpha + o(n^\alpha), & \text{if } d\ge 9.\end{cases} \tag{51}$$

Variance. We now prove the asymptotic estimate (4). First we have, by symmetry,

$$Q_{n,2} = 2^{d+1}\sum_{j_1+\cdots+j_{2^d}=n-1} \pi_{n,\mathbf{j}} M_{j_1,1}\left(M_{j_2,1}+\cdots+M_{j_{2^d},1}\right).$$

If $1\le d\le 8$, then the estimate (51) implies that $Q_{n,2} = O(n^{1-2\varepsilon})$. Thus a straightforward application of (12) yields

$$M_{n,2} = \mathbb{E}\left(\left(X_n - \mu_d n - \frac{\mu_d}{2^d-1}\right)^2\right) \sim \sigma_d^2 n,$$

which, by $\mathbb{V}(X_n) = M_{n,2} - M_{n,1}^2$ and (51), implies (4). Here σ_d^2 is given by

$$\sigma_d^2 = \frac{2}{d}\sum_{k,m\ge 0} \frac{V_k m! Q_{m,2}}{(k+2)\cdots(k+m+2)}, \tag{52}$$

with $Q_{0,2}$ and $Q_{1,2}$ properly defined. We will consider numeric evaluations of σ_d^2 later.

If $d\ge 9$, then, by (51),

$$Q_{n,2} = 2^{d+1}\sum_{j_1+\cdots+j_{2^d}=n-1} \pi_{n,\mathbf{j}}\left(c_+ j_1^{\alpha+i\beta} + K(\lambda_2) j_1^{\alpha-i\beta}\right) \times \sum_{2\le\ell\le 2^d}\left(c_+ j_\ell^{\alpha+i\beta} + K(\lambda_2) j_\ell^{\alpha-i\beta}\right) + o(n^{2\alpha}).$$

By the strong law of large numbers, we have

$$\begin{aligned}Q_{n,2} = 2^{d+1}\int_{[0,1]^d}\sum_{2\le\ell\le 2^d}\Big(&c_+^2 q_1(\mathbf{x})^{\alpha+i\beta}q_\ell(\mathbf{x})^{\alpha+i\beta}n^{2\alpha+2i\beta}\\&+c_+c_-\left(q_1(\mathbf{x})^{\alpha+i\beta}q_\ell(\mathbf{x})^{\alpha-i\beta}+q_1(\mathbf{x})^{\alpha-i\beta}q_\ell(\mathbf{x})^{\alpha+i\beta}\right)n^{2\alpha}\\&+c_-^2 q_1(\mathbf{x})^{\alpha-i\beta}q_\ell(\mathbf{x})^{\alpha-i\beta}n^{2\alpha-2i\beta}\Big)\,\mathrm{d}\mathbf{x}+o(n^{2\alpha}),\end{aligned}$$

where the $q_h(\mathbf{x})$'s are defined in (6). The integrals can be simplified as follows.

$$\begin{aligned}\eta(u,v) &:= \int_{[0,1]^d} q_1(\mathbf{x})^u \sum_{2\le\ell\le 2^d} q_\ell(\mathbf{x})^v\,\mathrm{d}x\\&=\sum_{0\le\ell<d}\binom{d}{\ell}\left(\frac{1}{u+v+1}\right)^{\ell}\left(\frac{\Gamma(u+1)\Gamma(v+1)}{\Gamma(u+v+2)}\right)^{d-\ell}\\&=\left(\frac{1}{u+v+1}+\frac{\Gamma(u+1)\Gamma(v+1)}{\Gamma(u+v+2)}\right)^{d}-\left(\frac{1}{u+v+1}\right)^{d},\end{aligned}\tag{53}$$

for $\Re(u),\Re(v)>-1$. Thus

$$\begin{aligned}Q_{n,2}2^{-d-1} &= c_+^2\eta(\alpha+i\beta,\alpha+i\beta)n^{2\alpha+2i\beta}+2c_-c_+\eta(\alpha+i\beta,\alpha-i\beta)n^{2\alpha}\\&\quad+c_-^2\eta(\alpha-i\beta,\alpha-i\beta)n^{2\alpha-2i\beta}+o(n^{2\alpha}).\end{aligned}$$

Transferring this approximation term by term using (15) gives

$$M_{n,2}=\tilde{G}_2(\beta\log n)n^{2\alpha}+o(n^{2\alpha}),$$

where

$$\begin{aligned}\tilde{G}_2(u) &:= 2^{d+1}c_+^2\eta(\alpha+i\beta,\alpha+i\beta)\frac{(2\alpha+2i\beta+1)^d}{P_0(2\alpha+2i\beta+1)}e^{2i\beta u}\\&\quad+2^{d+2}c_-c_+\eta(\alpha+i\beta,\alpha-i\beta)\frac{(2\alpha+1)^d}{P_0(2\alpha+1)}\\&\quad+2^{d+1}c_-^2\eta(\alpha-i\beta,\alpha-i\beta)\frac{(2\alpha-2i\beta+1)^d}{P_0(2\alpha-2i\beta+1)}e^{-2i\beta u}.\end{aligned}$$

This proves (4) with $G_2(x)=\tilde{G}_2(x)-G_1(x)^2$.

Asymptotic normality for $1\le d\le 8$. The same arguments used above for the variance also apply for $M_{n,k}$ for $k\ge 3$. By induction, we obtain

$$\begin{cases} M_{n,2k}\sim \dfrac{(2k)!}{2^k k!}\sigma_d^{2k}n^{2k};\\ M_{n,2k-1}=o(n^{k-1/2}),\end{cases}$$

for $k\ge 1$; details are omitted here for conciseness; see [3] for a similar proof. This proves the first part of Theorem 1.

Periodic fluctuations for $d \geq 9$**.** In this case, the same calculations for $\mathbb{V}(X_n)$ can be extended to show that

$$\mathbb{E}\left(\left(X_n - \mu_d n - \frac{\mu_d}{2^d - 1}\right)^k\right) \sim \tilde{G}_k(\beta \log n) n^{k\alpha} \qquad (k \geq 2); \tag{54}$$

where the $\tilde{G}_k$'s are bounded periodic functions. Then the proof that there is no fixed limit law for $(X_n - \mathbb{E}(X_n))/\sqrt{\mathbb{V}(X_n)}$ follows the same arguments used in [3].

Instead of giving the messy details of the proof for (54), we sketch the proof for

$$\|X_n - \mu_d n - 2\Re(n^{\alpha+i\beta} X)\|_p = o(n^\alpha) \qquad (p \geq 2), \tag{55}$$

where $\|Z\| = (\mathbb{E}|X|^p)^{1/p}$ denotes the usual L_p norm. Here X is a random variable with $\mathbb{E}(X) = c_+$ (see (49)) and defined by

$$X \stackrel{\mathscr{D}}{=} \langle U \rangle_1^{\alpha+i\beta} X^{(1)} + \cdots + \langle U \rangle_{2^d}^{\alpha+i\beta} X^{(2^d)},$$

where the $X^{(i)}$'s are independent copies of X and the $\langle U \rangle_i$'s are the volumes of the 2^d quadrants split by a random point in $[0,1]^d$. Part (ii) of Theorem 1 also follows from (55).

It suffices to prove $p = 2$, the remaining cases following by induction. The arguments used here are modified from those in [15] for random m-ary search trees.

Define

$$\begin{aligned}
\xi_n &:= \left\| X_n - \mu_d n - 2 \sum_{1 \leq j \leq 2^d} \Re\left(J_j^{\alpha+i\beta} X^{(j)}\right) \right\|_2, \\
\eta_n &:= \left\| 2 \sum_{1 \leq j \leq 2^d} \Re\left(J_j^{\alpha+i\beta} X^{(j)}\right) - 2 \sum_{1 \leq j \leq 2^d} \Re\left(n^{\alpha+i\beta} \langle U \rangle_j^{\alpha+i\beta} X^{(j)}\right) \right\|_2.
\end{aligned}$$

We prove that $\xi_n, \eta_n = o(n^\alpha)$, which will then imply (55) for $p = 2$.

First by the decomposition

$$\xi_n \leq \|X_n - \mu_d n\|_2 + 2^{d+2} \|J_1^{\alpha+i\beta} X^{(1)}\|_2,$$

we deduce that $\xi_n = O(n^\alpha)$. Then by the recurrence (5), we have the inequality

$$\xi_n^2 \leq \sum_{1 \leq j \leq 2^d} \mathbb{E}\left(\xi_{J_j} + \eta_{J_j}\right)^2 + o(n^{2\alpha}).$$

This, together with the estimate

$$\eta_n \leq 2^{d+2} n^\alpha \|X^{(1)}\|_2 \left\| \left(\frac{J_1}{n}\right)^{\alpha+i\beta} - \langle U \rangle_2^{\alpha+i\beta} \right\|_2 = o(n^\alpha),$$

gives

$$\begin{aligned}
\xi_n^2 &\leq 2^d \sum_{0 \leq j < n} \pi_{n,j} \xi_j^2 + o(n^{2\alpha}) \\
&= o(n^{2\alpha}),
\end{aligned}$$

by the o-version of (46).

d	$\mu_d \approx$
2	0.47841 76043 57434 47533 79639 99504 60454 12547 97628
3	0.56850 70194 06572 68270 35257 03246 03680 11920 50021
4	0.63168 48783 52998 69050 68769 97892 90145 67365 77851
5	0.67906 23676 94926 62299 74554 08602 48628 92348 92646
6	0.71615 83294 69847 70674 65510 61878 16738 93088 58805
7	0.74609 46112 09331 64803 70711 94105 57503 99390 36451
8	0.77079 60778 85838 99509 15248 99261 83895 90393 54520
9	0.79152 59978 40106 48407 81034 62942 59540 22737 03660
10	0.80915 45900 27608 17078 62137 34456 57737 58997 15908

Table 2: Approximate numeric values of μ_d for $d = 2, \dots, 10$.

3.2 Numerics of μ_d and σ_d^2

We consider means of computing numerically the constants μ_d and σ_d^2.

Numerical values of μ_d. To compute the constants μ_d to high precision, one can use either (2) or (50) by the standard procedure: compute the first few terms exactly and estimate the remaining terms by their asymptotic behaviors.

An alternative procedure is described in the last section. Consider $\bar{f}(z) := f(z) - \sum_{2\le j<N} A_j z^j$ ($A_1 = B_1$ and $B_n = 0$ for $n \ge 2$) for a suitably large number N, say 50. Exact values of A_n can be easily computed by the exact expression (1) when n is small. Observe that

$$\vartheta(z\vartheta)^{d-1} \sum_{j\ge N} c_j z^j = \sum_{j\ge N-1} c'_j z^j.$$

Thus the right-hand side of the DE

$$\left(\vartheta(z\vartheta)^{d-1} - 2^d\right) \bar{f} = 2^d z - \left(\vartheta(z\vartheta)^{d-1} - 2^d\right) \sum_{2\le j<N} A_j z^j,$$

contains only monomials z^j with $N < j < N + d$. Then the new $B^*(s)$ is of order s^{-N} for large s, implying a better convergence rate for the series (16) since V_k remains the same and can be computed recursively. Then we need only compute the first few terms (10 for example) of the series (16) to give the required degree of precision. In this way, we obtain Table 3.2. Such a procedure is also useful for other constants such as σ_d^2.

Expressions for σ_d^2. We first derive more explicit expressions for $M_{n,2}$ in (52) before computing σ_d^2.

We start from the bivariate generating function $F(z, y) := \sum_{n\ge 0} \mathbb{E}(e^{X_n y}) z^n/n!$, which satisfies, by (5), the equation

$$\frac{\partial}{\partial z} F(z, y) = e^y - 1 + \int_{[0,1]^d} F(q_1(\mathbf{x})z, y) \cdots F(q_{2^d}(\mathbf{x})z, y) \,\mathrm{d}\mathbf{x}.$$

In particular, $F(z, 0) = e^z$.

Then the Poisson generating function

$$\tilde{F}(z,y) = e^{-z}\sum_{n\geq 0} M_n(y)\frac{z^n}{n!} = e^{-z}\sum_{n\geq 0}\mathbb{E}(e^{(X_n-\mu_d n-\mu_d/(2^d-1))y})\frac{z^n}{n!}$$

satisfies the equation

$$\tilde{F}(z,y) + \frac{\partial}{\partial z}\tilde{F}(z,y) = e^{-z}(e^y-1)e^{-2^d\mu_d y/(2^d-1)} + \int_{[0,1]^d}\tilde{F}(q_1(\mathbf{x})z,y)\cdots\tilde{F}(q_{2^d}(\mathbf{x})z,y)\,\mathrm{d}\mathbf{x}.$$

Let $\tilde{F}(z,y) = \sum_{j\geq 0}\tilde{F}_j(z)y^j/j!$. Then

$$\tilde{F}_1'(z) + \tilde{F}_1(z) = e^{-z} + 2^d\int_{[0,1]^d}\tilde{F}_1(x_1\cdots x_d z)\,\mathrm{d}\mathbf{x},$$

with the initial condition $\tilde{F}_1(0) = -\mu_d/(2^d-1)$. The coefficients $u_n := n![z^n]\tilde{F}_1(z)$ satisfy

$$u_{n+1} + u_n = (-1)^n + \frac{2^d}{(n+1)^d}\,u_n,$$

which, after iterating, can be solved to be

$$u_n = (-1)^{n-1}\sum_{2\leq k\leq n}\prod_{k<\ell\leq n}\left(1-\frac{2^d}{\ell^d}\right) = (-1)^{n-1}[n]!\sum_{2\leq k\leq n}\frac{1}{[j]!},$$

for $n\geq 2$, with $u_0 = -\mu_d/(2^d-1)$ and $u_1 = 1-\mu_d$.

For $\tilde{F}_2(z)$, we have the same type of equation

$$\tilde{F}_2'(z) + \tilde{F}_2(z) = \tilde{g}_2(z) + 2^d\int_{[0,1]^d}\tilde{F}_2(x_1\cdots x_d z)\,\mathrm{d}\mathbf{x},$$

with the initial condition $\tilde{F}_2(0) = \mu_d^2/(2^d-1)^2$, where

$$\tilde{g}_2(z) := \left(1-\frac{2^{d+1}\mu_d}{2^d-1}\right)e^{-z} + 2^d\int_{[0,1]^d}\tilde{F}_1(x_1\cdots x_d z)\sum_{2\leq\ell\leq 2^d}\tilde{F}_1(q_\ell(\mathbf{x})z)\,\mathrm{d}\mathbf{x}. \tag{56}$$

Observe that

$$n![z^n]2^d\int_{[0,1]^d}\tilde{F}_1(x_1\cdots x_d z)\sum_{2\leq\ell\leq 2^d}\tilde{F}_1(q_\ell(\mathbf{x})z)\,\mathrm{d}\mathbf{x} = 2^d\sum_{0\leq j\leq n}\binom{n}{j}u_j u_{n-j}\eta(j,n-j),$$

where $\eta(j,n-j)$ is defined in (53).

By (56), we then have for $n\geq 0$

$$v_n := n!(-1)^n[z^n]\tilde{g}_2(z) = 1-\frac{2^{d+1}\mu_d}{2^d-1} + 2^d(-1)^n\sum_{0\leq j\leq n}\binom{n}{j}u_j u_{n-j}\eta(j,n-j).$$

It follows that

$$n![z^n]\tilde{F}_2(z) = (-1)^{n-1}[n]!\sum_{1\leq k<n}\frac{v_k}{[k+1]!}\qquad (n\geq 2),$$

d	$\sigma_d^2 \approx$
2	0.06145 73978 66984 07284 36701 54743 66750 63784
3	0.06802 65800 83909 72781 61723 15284 91262 75906
4	0.07090 19719 94546 02309 70950 30497 53882 55032
5	0.07261 12472 86535 68765 26637 38060 39503 98071
6	0.07449 21253 93111 00674 61761 51696 97039 29930
7	0.07731 76983 93655 71830 91768 87307 89088 95507
8	0.08123 98836 52827 96294 47650 19430 64044 32562

Table 3: Approximate numeric values of σ_d^2 for $d = 2, \dots, 8$. Note that $\sigma_1^2 = 2/45 \approx 0.04444\dots$.

with $\tilde{F}_2(0) = \mu_d^2/(2^d-1)^2$ and $\tilde{F}_2'(0) = 1 - 2^{d+1}\mu_d/(2^d-1) + (2^d+1)\mu_d^2/(2^d-1)$, and consequently

$$\begin{aligned} M_{n,2} &= \mathbb{E}\left(X_n - \mu_d\left(n + \frac{1}{2^d-1}\right)\right)^2 \\ &= \frac{\mu_d^2}{(2^d-1)^2} + \left(1 - \frac{2^{d+1}\mu_d}{2^d-1} + \frac{2^d+1}{2^d-1}\mu_d^2\right)n - \sum_{2\le k\le n}\binom{n}{k}(-1)^k[k]!\sum_{1\le j<k}\frac{v_j}{[j+1]!}. \end{aligned}$$

This provides a less dimension dependent expression for computing $M_{n,2}$ for small values of n needed for computing the approximate values of σ_d^2 in Table 3.2.

Note that for $1 \le d \le 8$, $M_{n,1} = O(n^{0.42})$ and

$$\mathbb{V}(X_n) = M_{n,2} - M_{n,1}^2 = \mathbb{E}\left(X_n - \mu_d\left(n + \frac{1}{2^d-1}\right)\right)^2 - M_{n,1}^2;$$

Thus to compute the limiting constant σ_d^2 of $\mathbb{V}(X_n)/n$, it suffices to compute $M_{n,2}$.

By the same procedure for computing μ_d, we obtain Table 3.2.

Note that

$$Q_{n,2} = [z^{n-1}]e^z\tilde{g}_2(z) = \sum_{0\le j<n}\binom{n-1}{j}(-1)^j v_j \qquad (n \ge 1).$$

For consistency, we can define $Q_{0,2} := \mu_d^2/(2^d-1)^2$. Then $Q_{1,2} = v_0 = 1-2^{d+1}\mu_d/(2^d-1)+2^d\mu_d^2/(2^d-1)$ and for $n \ge 2$

$$Q_{n,2} = 2^d\sum_{0\le m<n}\binom{n-1}{m}\sum_{0\le j\le m}\binom{m}{j}u_j u_{m-j}\eta(j, m-j).$$

3.3 Phase change of other cost measures

Consider the random variables defined recursively by

$$Y_n \overset{\mathscr{D}}{=} Y_{J_1}^{(1)} + \cdots + Y_{J_{2^d}}^{(2^d)} + T_n \qquad (n \ge 1), \tag{57}$$

with Y_0 given, where the $(Y_n^{(i)})$'s are independent copies of Y_n and T_n is a known random variable (often called "toll function").

3.3.1 Phase change of general toll functions

Our method of proof extends easily to cover a wide class of toll functions. We formulate a simple result for deterministic toll functions as follows.

Theorem 3. *If $T_n = O(n^{1/2}(\log n)^{-1/2-\varepsilon})$ and T_n is not identically 1 for all $n \geq 1$, then*

$$\frac{Y_n - \mu'_d n}{\sigma'_d \sqrt{n}} \xrightarrow{\mathscr{D}} N(0,1),$$

for $1 \leq d \leq 8$, where μ'_d and σ'_d are constants; if $d \geq 9$, then the sequence of random variables $(Y_n - \mathbb{E}(Y_n))/\sqrt{\mathbb{V}(Y_n)}$ does not converge to a fixed limit law.

The proof follows from that for Theorem 1 and is omitted. Both constants μ'_d and σ'_d can be computed by the same procedure as for μ_d and σ_d.

By the recurrence

$$\mathbb{V}(Y_n) = \sum_{0 \leq j < n} \pi_{n,j} \left(\mathbb{E}(Y_{j_1}) + \cdots + \mathbb{E}(Y_{j_{2^d}}) - \mathbb{E}(Y_n) + T_n\right)^2 + 2^d \sum_{0 \leq j < n} \pi_{n,j} \mathbb{V}(Y_j),$$

we see that the variance is identically zero iff $T_n \equiv 1$ for $n \geq 1$. In this case, $Y_n \equiv n$ (the total number of nodes in the tree). This also implies, when applying (12), the identity

$$\frac{2}{d} \sum_{k \geq 0} \frac{V_k}{(k+1)(k+2)} = 1 \qquad (d \geq 1). \tag{58}$$

The same method of proof we used for proving Theorem 1 also applies to cover the case when $T_n \sim \sqrt{n}$, which still leads to asymptotic normality for Y_n when $1 \leq d \leq 8$ with linear mean but with variance of order $n \log n$. The same non-existence of fixed limit law also holds in the wider range $T_n = o(n^\alpha)$ when $d \geq 9$. More cases can be clarified as in [7]. Since the number of concrete examples (directly related to cost measures of algorithms or quadtrees) is limited, we stop from considering other general limit results.

3.3.2 Concrete examples and extensions

We briefly discuss instead a few instances of T_n studied before in the literature.

Paging. The page usage of random quadtrees was studied in [26] and [19]; it can be regarded as a generalization of the number of leaves and satisfies (57) with $T_n = 1$ when $n > b$, and $T_n = 0$ otherwise, where $b \geq 0$ is a predetermined structural constant. We can also view Y_n as enumerating the number of nodes x with subtree sizes rooted at x larger than b.

By Theorem 3, *the page usage in random quadtrees undergoes the same type of phase change (of limit laws) as the number of leaves*. The mean constant is given by

$$\mu'_d(b) = \frac{2}{d} \sum_{k \geq 0} \frac{(b+1)! V_k}{(k+1)(k+2)\cdots(k+b+2)}.$$

If $d = 2$, then (see (35))

$$\begin{aligned}\mu_2'(b) &= 12(b+1)!\sum_{k\ge 0}\frac{(k+1)!}{(k+3)(k+4)(k+b+2)!}\\ &= 12(b+1)\int_0^1 (1-x)^b x^{-3}\left((1-x)\log(1-x)+x-\frac{x^2}{2}-\frac{x^3}{6}\right)\mathrm{d}x\\ &= 6b^2+9b+1-b(b+1)^2\pi^2+6b(b+1)^2\sum_{1\le j\le b} j^{-2},\end{aligned}$$

which coincides with the expression first derived in [26].

For $d \ge 3$, expressions for μ_d' are less explicit. We first simplify $\Upsilon(s)$ (see (40)) as follows.

$$\begin{aligned}\Upsilon(s) &= \sum_{k\ge 0}\frac{(b+1)!}{(k+2)\cdots(k+b+2)}\cdot\frac{\Gamma(k+1)\Gamma(1-s)}{\Gamma(k+2-s)}\\ &= (b+1)\sum_{0\le \ell\le b}\binom{b}{\ell}(-1)^\ell\Omega_{\ell+2}(s),\end{aligned}$$

where

$$\Omega_a(s) := \int_0^1 (1-x)^{-s}\sum_{k\ge 0}\frac{x^k}{k+a}\,\mathrm{d}x \qquad (\Re(s)<1;\ a=0,1,\dots),$$

(when $a = 0$, the term corresponding to $k = 0$ is dropped). Obviously, $\Omega_0(s) = (s-1)^{-2}$, and

$$\Omega_1(s) = \sum_{k\ge 1}\frac{1}{(s-k)^2} = \psi'(1-s) \qquad (\Re(s)<1).$$

By an integration by parts, we have the recurrence

$$\Omega_{a+1}(s) = \frac{s}{a}\,\Omega_a(s+1)+\frac{1}{a^2}-\frac{1}{as} \qquad (a\ge 1).$$

By induction

$$\Omega_a(s) = \binom{s+a-2}{a-1}\psi'(1-s)+\mathrm{poly}_1(a;s) \qquad (a=1,2,\dots),$$

where $\mathrm{poly}_1(a;s)$ is a polynomial of degree $a-2$ such that $\Omega_a(s)$ is of growth order $|s|^{-1}$ at infinity (with $|s-k|\ge\varepsilon$). More precisely, since

$$\psi'(1-s) = \sum_{j\ge 0}(-1)^{j+1}\mathbf{B}_j s^{-j-1} \qquad (|s|\to\infty, |\arg(-s)|\le\pi-\varepsilon),$$

where the $\mathbf{B}_j$'s denote Bernoulli numbers (see [14, p. 47, Eq. (7)])

$$\mathrm{poly}_1(a;s) = \sum_{1\le j<a} s^{j-1}\sum_{j\le \ell<a}\frac{|\mathbf{s}(a-1,\ell)|}{(a-1)!}(-1)^{j-\ell}\mathbf{B}_{j-\ell} \qquad (a\ge 2),$$

where the $\mathbf{s}(a-1,j)$'s denote Stirling numbers of the first kind. From this expression, we deduce the representation

$$\Upsilon(s) = \frac{(-1)^b}{b!}\,s(s-1)\cdots(s-b)\psi'(1-s)+\mathrm{poly}_2(b;s),$$

where $\mathrm{poly}_2(b;s)$ is a polynomial of degree b such that $\Upsilon(s)$ is of growth order $|s|^{-1}$ at infinity (with $|s-k| \geq \varepsilon$).

Then the integrand in the integral (40) has simple poles at $s = 1, 2, \dots, b$ and double poles at $s = b+1, b+2, \dots$. Summing over all residues of the poles yields

$$\mu_d'(b) = \frac{2^{d+1}}{d} \sum_{1 \leq k \leq b} \frac{(-1)^k}{\binom{b}{k}(k+1)^d k[k+1]!} + \frac{2^{d+1}}{d} \sum_{k > b} \frac{(-1)^b (b+1)\binom{k}{b+1}}{(k+1)^d k[k+1]!} \left(\sum_{1 \leq j < d} (\psi(k+2-\lambda_j) - \psi(k+1)) - \psi(k+1) + \psi(k-b) \right).$$

Note that the last series diverges for $b \geq d$. Numerically, the procedure we used for computing μ_d is preferable.

When $b \geq d$, we can use the recurrence

$$\mu_d'(b) = 2^{-d} \sum_{0 \leq j \leq d} R_{d,j} \mu_d'(b+j-1) \qquad (b \geq 1), \tag{59}$$

so that once the values $\{\mu_d'(0), \dots, \mu_d'(d-1)\}$ are known, all values of $\mu_d'(b)$ for higher values of b can be computed successively. Here $R_{d,j}$ is defined recursively as $R_{0,0} := 1$ and

$$R_{d,j} = (b+j+1)R_{d-1,j} - (b+j-1)R_{d-1,j-1} \qquad (0 \leq j \leq d), \tag{60}$$

with $R_{d,j} = 0$ when $j < 0$ or $j > d$. The recurrence (59) is proved using the DE (37) and successive integration by parts as follows.

$$\begin{aligned} \mu_d'(b) &= \frac{2}{d} \int_0^1 (1-x)^{b+1} V(x) \, \mathrm{d}x \\ &= \frac{2^{1-d}}{d} \int_0^1 \frac{(1-x)^b}{x^2} \left(x(1-x)\mathbb{D}\right)^d x^2 V(x) \, \mathrm{d}x \\ &= \frac{2^{1-d}}{d} \int_0^1 (1-x)^b R_d(x) V(x) \, \mathrm{d}x, \end{aligned}$$

where $R_d(x) = R_d(b;x)$ is defined by

$$\begin{aligned} R_d(x) &:= \frac{x^2}{(1-x)^b} \left(-\mathbb{D}x(1-x)\right)^d \frac{(1-x)^b}{x^2} \\ &= \sum_{0 \leq j \leq d} R_{d,j} (1-x)^j, \end{aligned}$$

with $R_{d,j}$ satisfying (by induction) the recurrence (60). Thus (59) follows. Note that when $b = 0$

$$\mu_d'(0) = \frac{2}{d} \int_0^1 (1-x) V(x) \, \mathrm{d}x = \frac{2^{1-d}}{d} \int_0^1 V(x) \, \mathrm{d}x = 1,$$

which can be proved directly by (40); see also (58).

Node sorts. If T_n is equal to the probability that the root has b nonempty subtrees, where $0 \le b \le 2^d$, then Y_n represents the number of nodes in random quadtrees having exactly b nonempty subtrees. The same type of phase change phenomenon holds since the toll function is bounded; see [34, 35] for expressions for the probability the root having b subtrees.

In general, if $T_n = \delta_{n,b}$, where $b \ge 0$, then the limits $\mu'_d = \mu'_d(b)$ of $\mathbb{E}(Y_n)/n$ are called *universal constants* in [36] since for general toll functions T_n with linear mean the linearity constant can be expressed in terms of the $\mu'_d(b)$'s as $\sum_{b\ge1} T_b \mu'_d(b)$. Expressions for $\mu'_d(b)$ can be derived similar to the previous case. We have

$$\begin{aligned}
\Upsilon(s) = \Upsilon_b(s) &= \sum_{k\ge0} \frac{b!\Gamma(k+1)\Gamma(1-s)}{(k+2)\cdots(k+b+2)\Gamma(k+2-s)} \\
&= -\sum_{0\le\ell\le b} \binom{b}{\ell} (-1)^\ell (\ell+1)\Omega_{\ell+2}(s) \\
&= (-1)^{b+1} \frac{s^2(s-1)\cdots(s-b+1)}{b!} \psi'(1-s) + \mathrm{poly}_3(b;s),
\end{aligned}$$

where $\mathrm{poly}_3(b;s)$ is a polynomial of degree b such that $\Upsilon(s)$ is of growth order $|s|^{-1}$ at infinity (with $|s-k| \ge \varepsilon$). Also $\mu'_d(b)$ satisfies the recurrence

$$\mu'_d(b) = 2^{-d} \sum_{0\le j\le d} R_{d,j} \mu'_d(b+j-1) \qquad (b \ge 1),$$

with $R_{d,j}$ satisfying $R_{d,j} = (b+j)R_{d-1,j} - (b+j-1)R_{d-1,j-1}$ for $0 \le j \le d$. Note that in this case $R_{d,0} = b^d$ and $R_{d,j} = (-1)^{d-1}(P_{j-1}(-b) - P_j(-b))$ for $1 \le j \le d$.

Total path length. In this case, $T_n = n-1$. Although Theorem 3 does not apply, our method of moments does, and we obtain convergence of all moments of $(Y_n - \mathbb{E}(Y_n))/n$ to some non-normal limit law for each $d \ge 1$; see [40], and [30] for similar details. In particular, the mean satisfies (see (14))

$$\mathbb{E}(Y_n) \sim \frac{2}{d} n \log n - \left(2 + \frac{2}{d} - 2\gamma - \frac{2}{d} \sum_{1\le j<d} \psi(2-\lambda_j)\right) n,$$

and the variance is asymptotic to $K_4 n^2$, where

$$K_4 = \frac{3^d}{3^d - 2^d} \int_{[0,1]^d} \left(1 + \frac{2}{d} \sum_{1\le j\le 2^d} q_j(\mathbf{x}) \log q_j(\mathbf{x})\right)^2 \mathrm{d}\mathbf{x}.$$

To evaluate the integral, let

$$\tilde\eta(u,v) = \int_{[0,1]^d} q_1(\mathbf{x})^u \sum_{1\le\ell\le2^d} q_\ell(\mathbf{x})^v \,\mathrm{d}\mathbf{x}.$$

Then $\tilde\eta(u,v) = \eta(u,v) + 1/(u+v+1)^d$, where η is defined in (53), so that

$$\tilde\eta(u,v) = \left(\frac{1}{u+v+1} + \frac{\Gamma(u+1)\Gamma(v+1)}{\Gamma(u+v+2)}\right)^d.$$

It follows that

$$K_4 = \frac{3^d}{3^d-2^d}\left(1+\frac{4}{d}\cdot\frac{\partial}{\partial v}\,\tilde\eta(0,v)\Big|_{v=1} + \frac{4}{d^2}\,2^d\frac{\partial^2}{\partial u\partial v}\,\tilde\eta(u,v)\Big|_{u=1,v=1}\right)$$
$$= \frac{3^d}{3^d-2^d}\cdot\frac{21-2\pi^2}{9d};$$

see also [40].

Unlike the number of leaves and other small cost measures, there is no change of limit law for total path length since the order of the variance is not alterned for increasing d.

Expected profiles (or depth). Denote by $Z_{n,k}$ the number of nodes at distance k to the root; the $Z_{n,k}$'s are informative shape characteristics often referred to as the *profiles* of the trees. The *depth* D_n is the distance of a randomly chosen node (all n nodes being equally likely) to the root. Then the probability that the depth is k equals $\mathbb{E}(Z_{n,k})/n$. Consider the level polynomials $L_n(y) := \sum_k \mathbb{E}(Z_{n,k})y^k$. Then $L_n(y)$ satisfies the recurrence

$$L_n(y) = 1 + 2^d y \sum_{0\le j<n} \pi_{n,j} L_j(y) \qquad (n \ge 1),$$

with $L_0(y) = 0$; see [19]. The same analysis for the small toll functions part of Theorem 2 (and the error analysis in Section 2.5) applies *mutatis mutandis* and yields

$$L_n(y) = \mathcal{K}(y)n^{2y^{1/d}-1} + O\left(n^{2\Re(y^{1/d}e^{2\pi i/d})-1} + n^{\varepsilon}\right), \tag{61}$$

where the O-term holds uniformly for y lying in some complex neighborhood of unity, and

$$\mathcal{K}(y) = \frac{2^d y^{1/d}}{d}\sum_{k\ge 2}\frac{\prod_{3\le\ell\le k}(1-2y^{1/d}/\ell)}{k^{d-1}\prod_{3\le\ell\le k}(1-2^d y/\ell^d)}\left((k-1)\sum_{1\le j<d}\left(\psi(k+1-\lambda_j y^{1/d})-\psi(k)\right)-1\right).$$

Thus the asymptotic normality (with optimal Berry-Esseen bound) of the depth D_n follows from (61) and the so-called quasi-power approximation theorems; see [24, Sec. IX.5] or [27]. Note that

$$\mathcal{K}(1) = \frac{2^{d+1}}{d}\sum_{k\ge 2}\frac{1}{k^d[k]!}\left(\sum_{1\le j<d}(\psi(k+1-\lambda_j)-\psi(k)) - \frac{1}{k-1}\right) = 1 \qquad (d\ge 2);$$

compare (58).

A considerable simplification of the expression for $\mathcal{K}(y)$ can be obtained by applying the finite difference integral representation for the closed-form expression (see [19])

$$L_n(y) = n - (1-y)\sum_{2\le k\le n}\binom{n}{k}(-1)^k\prod_{3\le j\le k}\left(1-\frac{2^d}{j^d}\,y\right) \qquad (n\ge 0),$$

giving

$$L_n(y) = -\frac{1}{2\pi i}\int_{\frac12-i\infty}^{\frac12+i\infty}\frac{\Gamma(n+1)\Gamma(-s)}{\Gamma(n+1-s)\Gamma(s+1)^d}\prod_{0\le\ell<d}\frac{\Gamma(s+1-\lambda_\ell y^{1/d})}{\Gamma(2-\lambda_\ell y^{1/d})}\,\mathrm{d}s.$$

Then, by moving the line of integration to the left and summing the simple poles encountered, we obtain

$$L_n(y) = \frac{1}{1-2^d y} + \mathcal{K}(y)n^{2y^{1/d}-1}\left(1+O\left(n^{-\varepsilon}+n^{-2\Re(y^{1/d}(1-e^{2\pi i/d}))}\right)\right),$$

uniformly for $|y| \geq 2^{-d} + \varepsilon$, where

$$\mathcal{K}(y^d) := \frac{1}{\Gamma(2y)^d(2y-1)} \prod_{1\leq \ell < d} \frac{\Gamma(2y(1-e^{2\ell\pi i/d}))}{\Gamma(2-2ye^{2\ell\pi i/d})}.$$

This explicit expression and the quasi-power theorems in [27] also give more precise estimates for the mean and variance of the depth

$$\begin{aligned}\mathbb{E}(D_n) &= \frac{2}{d}\log n + [t]\log \mathcal{K}(e^t) + o(1),\\ \mathbb{V}(D_n) &= \frac{2}{d^2}\log n + 2[t^2]\log \mathcal{K}(e^t) + o(1),\end{aligned}$$

where

$$\begin{aligned}[t]\log \mathcal{K}(e^t) &= K_2 - 1 = -2 - \frac{2}{d} + 2\gamma + \frac{2}{d}\sum_{1\leq j<d}\psi(2-\lambda_j),\\ 2[t^2]\log \mathcal{K}(e^t) &= \frac{2}{d}(1+\gamma) - \frac{2\pi^2}{3d} + \frac{2}{d^2} + \frac{2}{d^2}\sum_{1\leq j<d}\left(\psi(2-\lambda_j) + 2(1-\lambda_j)\psi'(2-\lambda_j)\right).\end{aligned}$$

Note that $n\mathbb{E}(D_n)$ equals the expected total path length, or A_n when $B_n = n-1$.

4 Second phase change: convergence rates and local limit theorems for X_n

We consider the convergence rate and local limit theorem for X_n, which undergo another phase change. Local limit theorems are more informative and precise than asymptotic normality. We use characteristic functions and standard Fourier analysis (see [42]), the main estimate needed being based on the refined method of moments introduced in [28] and the refined asymptotic transfers developed in Section 2.5.

Local limit theorems. To state our result, let

$$\bar{\alpha} := \begin{cases} 1/3, & \text{if } 1 \leq d \leq 7;\\ \sqrt{2}-1, & \text{if } d = 8.\end{cases}$$

Theorem 4. *Uniformly for* $x = o(n^{1/2-\bar{\alpha}})$,

$$\mathbb{P}\left(X_n = \left\lfloor X_n + x\sqrt{\mathbb{V}(X_n)}\right\rfloor\right) = \frac{e^{-x^2/2}}{\sqrt{2\pi\mathbb{V}(X_n)}}\left(1 + O\left((1+|x|^3)n^{-3(1/2-\bar{\alpha})}\right)\right).$$

The error terms in both cases are, up to the implied constants, optimal. Numerically, $3(1/2-\bar{\alpha}) \approx 0.2573$ when $d = 8$. This local limit theorem (in the range of moderate deviations) also implies the following convergence rate

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}} < x\right) - \Phi(x)\right| = \begin{cases} O(n^{-1/2}), & \text{if } 1\leq d\leq 7;\\ O(n^{-3(3/2-\sqrt{2})}), & \text{if } d = 8,\end{cases} \tag{62}$$

where $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-t^2/2}\,\mathrm{d}t$.

Moment generating function of X_n normalized by that of a normal distribution with the same mean and variance. Let $\Pi_n(y) := \mathbb{E}(e^{X_n y})$ and $\phi_n(y) := e^{-\mathbb{E}(X_n)y - \mathbb{V}(X_n)y^2/2}\Pi_n(y)$. From the recurrence (5), we have

$$\phi_n(y) = \sum_{j_1+\cdots+j_{2^d}=n-1} \pi_{n,\mathbf{j}}\phi_{j_1}(y)\cdots\phi_{j_{2^d}}(y)e^{\Delta_{n,\mathbf{j}}y+\nabla_{n,\mathbf{j}}y^2} \qquad (n \geq 1),$$

with $\phi_0(y) = 1$, where

$$\begin{aligned}\Delta_{n,\mathbf{j}} &= \delta_{n,1} + \mathbb{E}(X_{j_1}) + \cdots + \mathbb{E}(X_{j_{2^d}}) - \mathbb{E}(X_n),\\ \nabla_{n,\mathbf{j}} &= \frac{1}{2}\left(\mathbb{V}(X_{j_1}) + \cdots + \mathbb{V}(X_{j_{2^d}}) - \mathbb{V}(X_n)\right).\end{aligned}$$

Note that $\phi_n(y)$ is in general not a moment generating function.

Recurrences. Define $\phi_{n,k} := \phi_n^{(k)}(0)$. Then by the recurrence of $\phi_n(y)$, we have

$$\phi_{n,k} = \psi_{n,k} + 2^d \sum_{0\leq j<n} \pi_{n,j}\phi_{j,k} \qquad (n \geq 1),$$

where $\phi_{0,k} = 0$ and

$$\psi_{n,k} = \sum_{\substack{i_0+i_1+\cdots+i_{2^d}+2i_{2^d+1}=k\\ 0\leq i_1,\ldots,i_{2^d}<k}} \frac{k!}{i_0!\cdots i_{2^d}!i_{2^d+1}!} \sum_{j_1+\cdots+j_{2^d}=n-1} \pi_{n,\mathbf{j}}\phi_{j_1,i_1}\cdots\phi_{j_{2^d},i_{2^d}}\Delta_{n,\mathbf{j}}^{i_0}\nabla_{n,\mathbf{j}}^{i_{2^d+1}}.$$

A uniform upper bound for $\phi_{n,k}$. Recall that $\bar{\alpha} = 1/3$ when $1 \leq d \leq 7$, and $\bar{\alpha} = \sqrt{2}-1$ when $d = 8$. We will prove, by an inductive argument, that

$$|\phi_{n,k}| \leq k!A^k n^{k\bar{\alpha}} \qquad (k, n \geq 0), \tag{63}$$

where A is a suitable constant that will be specified later. Note that (63) holds for $k = 0, 1, 2$.

An upper bound for $\Delta_{n,\mathbf{j}}$. By the estimate (49), we have

$$\Delta_{n,\mathbf{j}} = O(n^\alpha) = \begin{cases} O\left(n^{1/3-\varepsilon}\right), & \text{if } 1 \leq d \leq 7;\\ O\left(n^{\sqrt{2}-1}\right), & \text{if } d = 8,\end{cases} \tag{64}$$

uniformly for all tuples $(j_1, \ldots, j_{2^d})$.

An upper bound for $\nabla_{n,\mathbf{j}}$. We need to refine the asymptotic estimate (4). Since the variance satisfies the recurrence

$$\mathbb{V}(X_n) = \sum_{j_1+\cdots+j_{2^d}=n-1} \pi_{n,\mathbf{j}}\Delta_{n,\mathbf{j}}^2 + 2^d \sum_{0\leq j<n} \pi_{n,j}\mathbb{V}(X_j),$$

and the first sum on the right-hand side is bounded above by

$$\sum_{j_1+\cdots+j_{2^d}=n-1} \pi_{n,\mathbf{j}}\Delta_{n,\mathbf{j}}^2 = \begin{cases} O\left(n^{2/3-2\varepsilon}\right), & \text{if } 1 \leq d \leq 7;\\ O\left(n^{2\sqrt{2}-2}\right), & \text{if } d = 8,\end{cases}$$

we obtain, by applying Corollary (1),

$$\mathbb{V}(X_n) = \sigma_d^2 n + \begin{cases} O\left(n^{2/3-2\varepsilon}\right), & \text{if } 1 \le d \le 7; \\ O\left(n^{2(\sqrt{2}-1)}\right), & \text{if } d = 8. \end{cases}$$

This implies that

$$\nabla_{n,\mathbf{j}} = \begin{cases} O\left(n^{2/3-2\varepsilon}\right), & \text{if } 1 \le d \le 7; \\ O\left(n^{2(\sqrt{2}-1)}\right), & \text{if } d = 8. \end{cases} \tag{65}$$

An estimate for $\phi_{n,3}$. From (64) and (65), it follows that

$$\psi_{n,3} = \begin{cases} O\left(n^{1-\varepsilon}\right), & \text{if } 1 \le d \le 7; \\ O\left(n^{3(\sqrt{2}-1)}\right), & \text{if } d = 8. \end{cases}$$

Thus (63) holds for $k = 3$ by applying (12) when $1 \le d \le 7$ and (15) when $d = 8$.

Induction. For higher values of k, we use the estimates (by (64) and (65))

$$|\Delta_{n,\mathbf{j}}| \le K_5 n^{\bar{\alpha}}, \quad |\nabla_{n,\mathbf{j}}| \le K_6 n^{2\bar{\alpha}}, \tag{66}$$

uniformly for all tuples $(j_1, \dots, j_{2^d})$.

Assume that (63) holds $\phi_{n,i}$ for $i < k$. Then by (66) and induction

$$\begin{aligned} |\psi_{n,k}| &\le k! n^{k\bar{\alpha}} \sum_{\substack{i_0+\cdots+i_{2^d}+2i_{2^d+1}=k \\ 0 \le i_1,\dots,i_{2^d}<k}} A^{i_1+\cdots+i_{2^d}} \frac{K_5^{i_0} K_6^{i_{2^d+1}}}{i_0! i_{2^d+1}!} \sum_{j_1+\cdots+j_{2^d}=n-1} \pi_{n,\mathbf{j}} \left(\frac{j_1}{n}\right)^{i_1\bar{\alpha}} \cdots \left(\frac{j_{2^d}}{n}\right)^{i_{2^d}\bar{\alpha}} \\ &\le k! n^{k\bar{\alpha}} e^{K_5+K_6} \sum_{0 \le \ell \le k} A^\ell S(\ell), \end{aligned} \tag{67}$$

where

$$S(\ell) := \sum_{i_1+\cdots+i_{2^d}=\ell} \sum_{j_1+\cdots+j_{2^d}=n-1} \pi_{n,\mathbf{j}} \left(\frac{j_1}{n}\right)^{i_1\bar{\alpha}} \cdots \left(\frac{j_{2^d}}{n}\right)^{i_{2^d}\bar{\alpha}}.$$

An estimate for $S(\ell)$. We now show that $S(\ell) \to 0$ as $\ell \to \infty$.

Lemma 5. *For $\ell \ge 0$*

$$S(\ell) \le c(\ell\bar{\alpha} + 1)^{-d} \qquad (d \ge 1), \tag{68}$$

where $c > 0$ is independent of ℓ and n.

Proof. First, by the strong law of large numbers

$$\begin{aligned} S(\ell) &\le c \int_{[0,1]^d} \sum_{i_1+\cdots+i_{2^d}=\ell} \prod_{1 \le h \le 2^d} q_h(\mathbf{x})^{i_h\bar{\alpha}} \mathrm{d}\mathbf{x} \\ &= c 2^d [z^\ell] \int_{[0,1/2]^d} \prod_{1 \le h \le 2^d} \frac{1}{1 - q_h(\mathbf{x})^{\bar{\alpha}} z} \mathrm{d}\mathbf{x}. \end{aligned}$$

Observe that the smallest term among the $q_h(\mathbf{x})$'s is $q_{2^d}(\mathbf{x}) = (1-x_1)\cdots(1-x_d)$ when $\mathbf{x} \in [0,1/2]^d$. Thus the dominant term for large ℓ comes from $q_{2^d}(\mathbf{x})$, and it follows that

$$\begin{aligned}
[z^\ell]\int_{[0,1/2]^d} \prod_{1\le h\le 2^d} \frac{1}{1-q_h(\mathbf{x})^{\bar\alpha} z}\,\mathrm{d}\mathbf{x} &\sim \int_{[0,1/2]^d} q_{2^d}(\mathbf{x})^{\ell\bar\alpha} \prod_{1\le h<2^d} \frac{1}{1-q_h(\mathbf{x})/q_{2^d}(\mathbf{x})}\,\mathrm{d}\mathbf{x}\\
&\sim \int_{[0,1/2]^d} (1-x_1)^{\ell\bar\alpha}\cdots(1-x_1)^{\ell\bar\alpha}\,\mathrm{d}\mathbf{x}\\
&\sim (\ell\bar\alpha+1)^{-d}.
\end{aligned}$$

This proves (68). ∎

Proof of (63). Substituting the estimate (68) into (67), we obtain

$$|\psi_{n,k}| \le \frac{c}{(k\bar\alpha+1)^d} k! A^k n^{k\bar\alpha}.$$

Then, by the asymptotic transfer (15),

$$|\phi_{n,k}| \le \frac{c'}{(k\bar\alpha+1)^d} k! A^k n^{k\bar\alpha},$$

where c' is independent of n and k. Thus $c'/(k\bar\alpha+1)^d < 1$ for large enough k, say $k \ge k_0$. Hence, (63) follows by suitably tuning A for $k \le k_0$; see [1] for similar details.

An estimate for the characteristic function for small y. Denote by $\varphi_n(y) = \Pi_n(iy/\sqrt{\mathbb{V}(X_n)})$. Then, by (63) and the Taylor series expansion,

$$\left|\varphi_n(y) - e^{-y^2/2}\right| = O\left(|y|^3 n^{-3(1/2-\bar\alpha)} e^{-y^2/2}\right) \tag{69}$$

for $|y| \le \varepsilon_0 n^{1/2-\bar\alpha}$, where $\varepsilon_0 > 0$ is sufficiently small.

A uniform estimate for $\Pi_n(iy)$ for $|y| \le \varepsilon$. From (69), we deduce that

$$|\Pi_n(iy)| \le e^{-\varepsilon_1(n+1)y^2} \qquad (n \ge 3), \tag{70}$$

for $|y| \le \varepsilon_0 n^{-\bar\alpha}$, where ε_1 is a suitably chosen small constant.

We now prove that the estimate (70) indeed holds for $|y| \le \varepsilon_2$, $\varepsilon_2 > 0$ being a small constant. To that purpose, choose n_0 large enough and set $\varepsilon_2 := \varepsilon_0 n_0^{-\bar\alpha}$. Then, (70) holds for $3 \le n \le n_0$ and $|y| \le \varepsilon_2$. For $n > n_0$, by (5) and induction,

$$\begin{aligned}
|\Pi_n(iy)| &\le \sum_{j_1+\cdots+j_{2^d}=n-1} \pi_{n,\mathbf{j}} |\Pi_{j_1}(iy)|\cdots|\Pi_{j_{2^d}}(iy)|\\
&\le e^{-\varepsilon_1(n+1)y^2 - \varepsilon_1(2^d-2)y^2}\\
&\le e^{-\varepsilon_1(n+1)y^2}.
\end{aligned}$$

This concludes the induction proof.

Reformulating the estimate (70) yields the following global estimate for $\varphi_n(y)$

$$|\varphi_n(y)| = O\left(e^{-\varepsilon n y^2}\right) \qquad (n \ge 3), \tag{71}$$

uniformly for $|y| \le \varepsilon_2 n^{1/2}$.

Berry-Esseen bounds and local limit theorems. The convergence rates (62) now follows by (69), (71) and the Berry-Esseen smoothing inequality

$$\sup_x \left| \mathbb{P}\left(\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}} < x \right) - \Phi(x) \right| = O\left(R_n^{-1} + \int_{R_n}^{R_n} \left| \frac{\varphi_n(y) - e^{-y^2/2}}{y} \right| \mathrm{d}t \right),$$

where $R_n := \varepsilon n^{3(1/2-\bar{\alpha})}$; see [42].

For local limit theorems, we first observe that the span of X_n is 1 by induction, so that (70) can be extended to $|y| \le \pi$ (again by induction). Then Theorem 4 follows by applying the Fourier inversion formula

$$\mathbb{P}(X_n = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iky} \Pi_n(iy)\, \mathrm{d}y,$$

where $k = \left\lfloor \mathbb{E}(X_n) + x\sqrt{\mathbb{V}(X_n)} \right\rfloor$; see Figure 3.

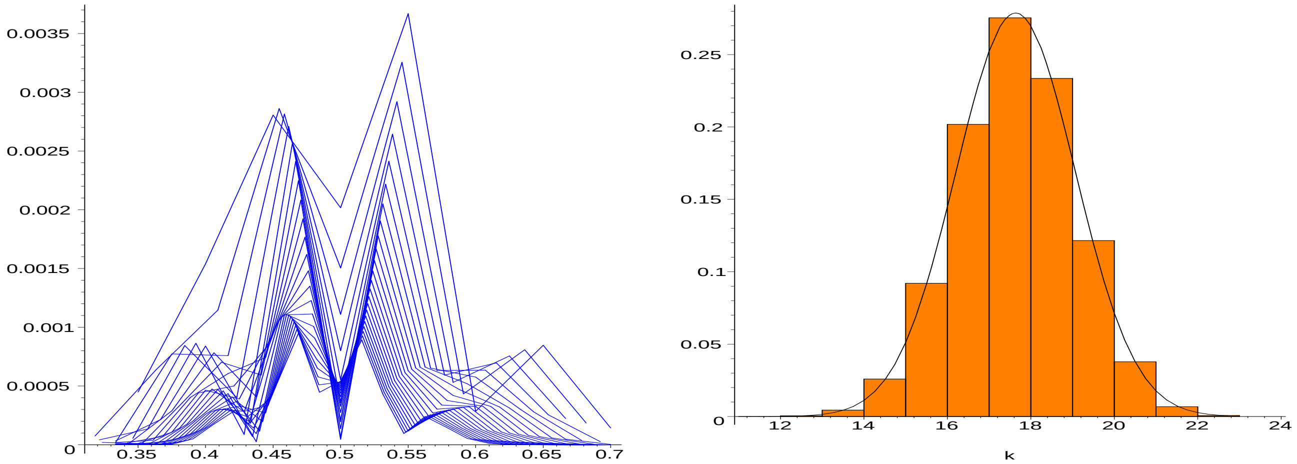

Figure 3: *Left: A Sedgewick plot of the absolute difference between* $\mathbb{P}(X_n = k)$ *and* $e^{-(k-\mathbb{E}(X_n))^2/(2\mathbb{V}(X_n))}/\sqrt{2\pi\mathbb{V}(X_n)}$ *for* $n = 20, 22, \dots, 64$ *and* $\lfloor 0.35n \rfloor \le k \le \lfloor 0.7n \rfloor$ *(normalized in the unit interval) when* $d = 2$*. Right: the histogram of* $\mathbb{P}(X_n = k)$ *for* $d = 3, n = 30$ *and* $k = 12, \dots, 23$*, together with the corresponding normal curve (having the same mean and variance).*

Extensions to general cost measures. The same method of proof applies to other cost measures in random quadtrees. In particular, Assume that T_n in (57) is deterministic and satisfies $T_n = O(n^\rho)$, where $\rho < 1/2$. If $1 \le d \le 7$, then we have the following Berry-Esseen bounds for Y_n.

$$\sup_x \left| \mathbb{P}\left(\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}} < x \right) - \Phi(x) \right| = \begin{cases} O(n^{-1/2}), & \text{if } \rho < 1/3; \\ O(n^{-1/2}\log n), & \text{if } \rho = 1/3; \\ O(n^{-3(1/2-\rho)}), & \text{if } 1/3 < \rho < 1/2. \end{cases}$$

When $d = 8$, then

$$\sup_x \left| \mathbb{P}\left(\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}} < x \right) - \Phi(x) \right|$$
$$= \begin{cases} O(n^{-3(3/2-\sqrt{2})}), & \text{if } \rho < \sqrt{2} - 1; \\ O(n^{-3(3/2-\sqrt{2})}(\log n)^3), & \text{if } \rho = \sqrt{2} - 1; \\ O(n^{-3(1/2-\rho)}), & \text{if } \sqrt{2} - 1 < \rho < 1/2. \end{cases}$$

The corresponding local limit theorems can be derived when Y_n assumes only integer values.

5 Random d-dimensional grid-trees

We consider briefly the phase changes in random grid-trees in this section, the required asymptotic transfers being also given.

Grid trees. Devroye [12] extended the d-dimensional point quadtrees and m-ary search trees as follows. Instead of choosing the first point as the root, one chooses, say the first $m - 1$ points ($m \geq 2$) and places them at the root. These $m - 1$ points then split the space into m^d smaller regions (called grids) when no pair of points is collinear. Each node in the corresponding grid-tree has at most m^d subtrees. When $m = 2$, grid-trees are quadtrees; when $d = 1$, grid-trees reduce to the usual m-ary search trees; see [37].

Random grid-trees. Fix $m \geq 2$ and $d \geq 1$ throughout this section. Assume that the input is a sequence of n random points uniformly and independently chosen from $[0, 1]^d$. Construct the grid-tree from this sequence. The resulting tree is called a *random grid-tree*.

Phase changes of the number of leaves. For simplicity of presentation, we consider the number of leaves in random grid-trees, denoted by X_n.

m	2	3	4	$5, \ldots, 8$	$9, \ldots, \underline{26}$
d	$1, \ldots, \underline{8}$	$1, \ldots, 4$	$1, \ldots, 3$	$1, 2$	1

Table 4: The set $\mathcal{S}$ of all pairs of (m, d) for which X_n is asymptotically normally distributed. The two boundary cases $(2, 26)$ (m-ary search trees) and $(1, 8)$ (quadtrees) are both underlined.

Theorem 5. *If $(m, d) \in \mathcal{S}$, where $\mathcal{S}$ is given in Table 4, then*

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}} \xrightarrow{\mathscr{M}} N(0, 1);$$

if $m \geq 2, d \geq 1$ and $(m, d) \notin \mathcal{S}$, then the sequence of random variables $(X_n - \mathbb{E}(X_n))/\sqrt{\mathbb{V}(X_n)}$ does not converge to a fixed limit law.

More refined results (and more phase changes) can be derived as in the case of quadtrees.

Recurrence of X_n. The recurrence of X_n now has the form

$$X_n \stackrel{\mathscr{D}}{=} \sum_{1\le j\le m^d} X^{(j)}_{J_j} + \delta_{n,1}, \qquad (n \ge 1),$$

with $X_0 = 0$, where $X_n, X_n^{(1)}, \dots, X_n^{(m^d)}, (J_1, \dots, J_{m^d})$ are independent and $X_n \stackrel{\mathscr{D}}{=} X_n^{(j)}, 1 \le j \le m^d$. Moreover, the splitting probabilities can be expressed as

$$\begin{aligned}\pi_{n,\mathbf{j}} &= \mathbb{P}\left(J_1 = j_1, \dots, J_{m^d} = j_{m^d}\right)\\ &= \binom{n-m+1}{j_1, \dots, j_{m^d}} \int_{([0,1]^d)^{m-1}} \prod_{\substack{1\le h\le m^d\\ h-1=(b_1,\dots,b_d)_m}} q_h(\mathbf{x}_1, \dots, \mathbf{x}_{m-1})^{j_h} \mathrm{d}\mathbf{x}_1 \dots \mathrm{d}\mathbf{x}_{m-1}\end{aligned}$$

for all $j_1 + \cdots + j_{m^d} = n - m + 1$, where

$$q_h(\mathbf{x}_1, \dots, \mathbf{x}_{m-1}) = \prod_{1\le i\le d} \sum_{0\le \ell<m} 1_{\{\ell\}}(b_i) \left(x^{(i)}_{(\ell+1)} - x^{(i)}_{(\ell)}\right),$$

with $x_{(\ell)}$ denoting the ℓ-th order statistic of $x_1, \dots, x_{m-1}$ ($x_{(0)} := 0, x_m := 1$).

Recurrence of moments. All moments satisfy recurrences of the form

$$A_n = B_n + m^d \sum_{0\le j\le n-m+1} \pi_{n,j} A_j, \qquad (n \ge m-1), \tag{72}$$

where $\pi_{n,j}$ denotes the probability that a specified subtree (say the first) of the root has j nodes.

We now show that $\pi_{n,j}$ can be expressed in the form

$$\pi_{n,j} = \sum_{j\le j_1\le\cdots\le j_{d-1}\le n-m+1} \frac{\binom{n-1-j_{d-1}}{m-2}}{\binom{n}{m-1}} \prod_{1\le i<d} \frac{\binom{j_i-j_{i-1}+m-2}{m-2}}{\binom{j_i+m-1}{m-1}}. \tag{73}$$

To that purpose, we first split $\pi_{n,j}$ as follows.

$$\pi_{n,j} = \sum_{j\le i_1\le i_2\le\cdots\le i_{d-1}\le n-m+1} \varpi_{j;i_1,\dots,i_{d-1}},$$

where $\varpi_{j;i_1,\dots,i_{d-1}}$ denotes the probability that the n random points are distributed in the d-dimensional unit cube in the following way: the first $m-1$ points, denoted by $\mathbf{x}_1, \dots, \mathbf{x}_{m-1}$, split $[0,1]^d$ into m^d grids and the remaining points are placed in these grids such that grids of the form

$$\left[0, x^{(1)}_{(1)}\right] \times \cdots \times \left[0, x^{(i)}_{(1)}\right] \times \left[x^{(i+1)}_{(1)}, 1\right] \qquad (i = 0, \cdots, d),$$

contain $n - m - i_{d-1} + 1, i_{d-1} - i_{d-2}, \dots, i_1 - j, j$ random points, respectively.

By definition, we have

$$\begin{aligned}\frac{\varpi_{j;i_1,\dots,i_{d-1}}}{\binom{n-m+1}{i_0,i_1-i_0,\dots,i_d-i_{d-1}}} &= \int_{([0,1]^d)^{m-1}} \prod_{1\le i\le d} \left(x^{(i)}_{(1)}\right)^{i_{d-i}} \left(1 - x^{(i)}_{(1)}\right)^{i_{d-i+1}-i_{d-i}} \mathrm{d}\mathbf{x}_1 \dots \mathrm{d}\mathbf{x}_{m-1}\\ &= \prod_{1\le r\le d} \int_{[0,1]^{m-1}} \left(x^{(r)}_{(1)}\right)^{i_{d-r}} \left(1 - x^{(r)}_{(1)}\right)^{i_{d-r+1}-i_{d-r}} \mathrm{d}x^{(r)}_1 \dots \mathrm{d}x^{(r)}_{m-1},\end{aligned} \tag{74}$$

where $i_0 := j$ and $i_d := n - m + 1$. It remains to evaluate integrals of the form

$$\int_{[0,1]^{m-1}} x_{(1)}^{\rho} \left(1 - x_{(1)}\right)^{\tau} \mathrm{d}x_1 \dots \mathrm{d}x_{m-1},$$

where $\rho, \tau \geq 0$. By dividing the domain of integration into $(m-1)!$ sets of the form $\{(x_1, \dots, x_{m-1}) | x_{\sigma(1)} < \cdots < x_{\sigma(m-1)}\}$, where σ runs through all permutations of $m-1$ elements

$$\begin{aligned} \int_{[0,1]^{m-1}} x_{(1)}^{\rho} \left(1 - x_{(1)}\right)^{\tau} \mathrm{d}x_1 \dots \mathrm{d}x_{m-1} &= (m-1)! \int_{0 \leq x_1 \leq \cdots \leq x_{m-1} \leq 1} x_1^{\rho} \left(1 - x_1\right)^{\tau} \mathrm{d}x_1 \dots \mathrm{d}x_{m-1} \\ &= (m-1) \int_0^1 x_1^{\rho} \left(1 - x_1\right)^{\beta + m - 2} \mathrm{d}x_1 \\ &= (m-1) \frac{\Gamma(\rho+1)\Gamma(\tau+m-1)}{\Gamma(\rho+\tau+m)}, \end{aligned}$$

by symmetry. Substituting this expression into (74) gives the desired result (73). ▮

The DE. Let $A(z) = \sum_{n \geq 0} A_n z^n$, $B(z) = \sum_{n \geq 1} B_n z^n$, and $f = A - B$. Then the recurrence (72) translates into the DE

$$(1-z)^{m-1} \mathbb{D}^{m-1} \left(z^{m-1}(1-z)^{m-1}\mathbb{D}^{m-1}\right)^{d-1} f(z) = m!^d A(z),$$

or, in terms of the ϑ-operator,

$$\vartheta^{\overline{m-1}} \left(z^{m-1} \vartheta^{\overline{m-1}}\right)^{d-1} f(z) = m!^d A(z), \tag{75}$$

where $\vartheta^{\overline{m-1}} = \vartheta(\vartheta+1)\cdots(\vartheta+m-2)$.

The normal form. We then rewrite the DE in the form

$$P_0(\vartheta) f(z) = \sum_{1 \leq j \leq (m-1)(d-1)} (1-z)^j P_j(\vartheta) f(z) + m!^d B(z),$$

where the P_j's are polynomials of degree dm. In particular,

$$P_0(\vartheta) = (\vartheta^{\overline{m-1}})^d - m!^d = \prod_{1 \leq j \leq d} \left(\vartheta^{\overline{m-1}} - m! e^{2j\pi i/d}\right).$$

The unique case when the above DE reduces to a pure Cauchy-Euler type is $d = 1$. Also the "linearization" achieved by the Euler transform does not seem to work directly for $m \geq 3$. This says that it is not obvious how to derive an explicit expression such as (38) when $m \geq 3$.

Zeros of $P_0(x)$. Our method of proof for deriving the asymptotic transfers is mostly operational and requires only limited properties of the zeros of the indicial polynomial $P_0(x)$. The proofs of the following properties are straightforward and thus omitted.

- The zero with the largest real part is $x = 2$. All other zeros have real parts strictly less than 2.
- All zeros of $P_0(x)$ are simple (we need only this property for $x = 2$ and the second largest zeros in real part).

Other properties similar to those for the case $d = 1$ (m-ary search trees) can be derived as in [37, Ch. 3].

Asymptotic transfers. We state the main asymptotic transfers needed for proving Theorem 5.

Let $H_m := \sum_{1\le j\le m} 1/j$ denotes the harmonic numbers. Define

$$K_B := \frac{1}{d(H_m-1)} \sum_{k\ge 0} V_k B^*(k+2), \tag{76}$$

when the series converges, where V_k is defined recursively by $V_k = 0$ when $k<0$, $V_0 = 1$, and

$$V_k = \sum_{1\le \ell\le (m-1)(d-1)} \frac{P_\ell(k+2)}{P_0(k+2)} V_{k-\ell} \qquad (k\ge 1),$$

and $B^*(s) := \int_0^1 B(x)(1-x)^{s-1}\,\mathrm{d}x$ when the integral converges.

Theorem 6. *Let A_n be defined by the recurrence (72) with A_0 and $\{B_n\}_{n\ge 1}$ given. Then*

(i) *(Small toll functions)*

$$A_n \sim K_B n \quad \textit{iff} \quad B_n = o(n) \ \textit{ and } \left|\sum_n B_n n^{-2}\right| < \infty,$$

where the constant K_B is given in (76);

(ii) *(Linear toll functions) Assume that $B_n = cn + u_n$, where $c\in\mathbb{C}$ and u_n is a sequence of complex numbers. Then*

$$A_n \sim \frac{c}{d(H_m-1)} n\log n + K_1 n \quad \textit{iff} \quad u_n = o(n) \textit{ and } \left|\sum_n u_n n^{-2}\right| < \infty.$$

Here $K_1 := cK_2 + K_u$ with K_u defined by replacing the sequence B_n by u_n in (76) and K_2 given explicitly by

$$K_2 := \frac{1}{d(H_m-1)}\left(\sum_{k\ge 1}\frac{V_k}{k(k+1)} + \gamma - 2 - \frac{d}{2}(H_m-1) + \frac{H_m^{(2)}-1}{2(H_m-1)}\right),$$

where $H_m^{(2)} := \sum_{1\le j\le m} 1/j^2$.

(iii) *(Large toll functions) Assume that $\Re(\upsilon) > 1$ and $c\in\mathbb{C}$. Then*

$$B_n \sim cn^{\upsilon} \quad \textit{iff} \quad A_n \sim \frac{c((\upsilon+1)^{\overline{m-1}})^d}{((\upsilon+1)^{\overline{m-1}})^d - m!^d} n^{\upsilon}.$$

In particular, if $d=1$, then $V_k = \delta_{k,0}$ and

$$K_B = \frac{B^*(2)}{H_m-1} = \frac{1}{H_m-1}\sum_{k\ge 0}\frac{B_k}{(k+1)(k+2)};$$

see [3].

Growth order of V_k for grid-trees. The sequence V_k satisfies the DE

$$\begin{aligned}\left((\mathbb{D}_z z+m-2)\cdots(\mathbb{D}_z z+1)\mathbb{D}_z z(1-z)^{m-1}\right)^{d-1}\\ \times(\mathbb{D}_z z+m-2)\cdots(\mathbb{D}_z z+1)\mathbb{D}_z\left(z^2V(z)\right)-m!^d zV(z)=0,\end{aligned}$$

implying that the solution of the form $V(z)=(1-z)^{-s}\phi(1-z)$ has the indicial equation

$$s^d(s+1)^d\cdots(s+m-2)^d=0.$$

Thus we deduce that

$$V_k=O\left(k^{-1}(\log k)^c\right),$$

for some $c\geq d-2$. This implies that the series in (76) is convergent for both cases of small and linear toll functions.

Refining the asymptotic transfer for small toll functions. To derive the second-order term for $\mathbb{E}(X_n)$ and $\mathbb{V}(X_n)$, we also need the following types of transfer.

Let $\alpha+1$ denote the real part of the second largest zeros of $P_0(x)$ (all zeros arranged in decreasing order according to their real parts), and $\beta>0$ denote the absolute value of the imaginary part of either zero.

Proposition 2. *Assume that A_n satisfies (72).*

(i) *If $B_n\sim cn^{\upsilon}$, where $c\in\mathbb{C}$ and $\alpha<\Re(\upsilon)<1$, then*

$$A_n=K_Bn+\frac{c((\upsilon+1)^{\overline{m-1}})^d}{((\upsilon+1)^{\overline{m-1}})^d-m!^d}n^{\upsilon}+o(n^{\upsilon}+n^{\varepsilon}),$$

where K_B is defined in (76).

(ii) *If $B_n=o(n^{\alpha})$, then*

$$A_n=K_Bn+K(\lambda_1)n^{\alpha+i\beta}+K(\lambda_2)n^{\alpha-i\beta}+o(n^{\alpha}+n^{\varepsilon}),$$

where the $K(\lambda_j)$'s are constants whose expressions are similarly defined as in (48). If the B_k's are all real, then $K(\lambda_1)=\overline{K(\lambda_2)}$.

These types of transfer and the inductive arguments used for quadtrees can be applied to prove local limit theorems for X_n with optimal convergence rates. Limit theorems for many other shape parameters can also be derived. We mention only the application to total path length.

Total path length. Neininger and Rüschendorf [40] derived a general limit law for the total path length in random split trees of Devroye (see [12]), which cover in particular grid-trees. Their result is based on the assumption that the expected total path length satisfies asymptotically $cn\log n+c'n$. Our asymptotic transfer for linear toll functions shows that this is the case for grid-trees. This proves the limit law for the total path length in random grid-trees. Note that the limit law can also be derived directly by method of moments and our asymptotic transfer for large toll functions.

6 Conclusions

We extended in this paper the asymptotic theory for Cauchy-Euler DEs developed in [7] to essentially DEs with polynomial coefficients (often referred to as *holonomic DEs*) and $z = 0$ not an irregular singularity. Not only the results are very general, but also the method of proof requires almost no knowledge on DEs. Indeed, since all our manipulations are based on linear operators, only properties of the first-order DEs are used, which can be further avoided by completely operating on recurrences of quicksort type (see [30]). The main feature of such an approach is that all differential operators are regarded as coefficient-transformers, so that no analytic properties are needed for the functions involved.

We applied the general asymptotic transfers developed in this paper to clarify the phase changes of limit laws in quadtrees and more general grid-trees. Further applications to distributional properties of profiles of random search trees will be given elsewhere.

For more methodological interest, we conclude this paper by mentioning an alternative approach to proving general asymptotic transfers for A_n (under suitable growth information on B_n) based solely on the theory of differential equations. Such an approach was inspired by the series of papers by Flajolet and his coauthors (see [17, 20, 22, 26]). We start from the method of Frobenius and seeks solutions of the form $(1-z)^{-\lambda_k}\phi(1-z)$ for the homogeneous DE $(\vartheta(z\vartheta)^{d-1} - 2^d)f(z) = 0$, where $\phi(z)$ is analytic at $z = 0$. A detailed information on the zeros of $P_0(x)$ is needed; in particular, we can show that when d is a multiple of 6 there are two pairs of non-real zeros differing by integers (in that case, logarithmic terms need to be introduced). Then we use the method of variation of parameters (see [32]) for the non-homogeneous DE; a long and laborious calculation of the Wronskians then leads to the form

$$\begin{aligned} f(z) = &\sum_{0\le j<d} \xi_j(z)(1-z)^{-\lambda_j} \\ &+ 2^d \sum_{0\le j<d} \eta_j(z)(1-z)^{-\lambda_j} \int_0^z (1-t)^{\lambda_j-1} B(t) \sum_{0\le r\le \kappa_d} \zeta_{j,r}(t) \left(\log\frac{z}{t}\right)^r \,\mathrm{d}t, \end{aligned} \tag{77}$$

where $\kappa_d \le (d-1)^2$ and $\xi_j, \eta_j, \zeta_{j,r}$ are functions analytic in the unit circle satisfying $\sum_n |[z^n]\chi(z)| < \infty$, where $\chi \in \{\xi_j, \eta_j, \zeta_{j,r}\}$. Similar expressions can be derived for $\sum_{1\le j<d}(1-z)^j P_j(\vartheta)f$. Then the sufficiency proofs of the transfers (12), (13), (15) are reduced to deriving asymptotic transfers for integrals of the form

$$\xi(z)(1-z)^{-\upsilon}\int_0^z (1-t)^{\upsilon-1}B(t)\eta(t)\left(\log\frac{z}{t}\right)^r \,\mathrm{d}t.$$

Such a general approach, although quickly gives the general form of the solution, does not seem easily amended for getting expressions for the leading constants (similar to most asymptotic problems on DEs and linear differential systems); also for more general DEs such as (75), the precise characterizetion of the zero locations (of their differences) requires more delicate analysis.

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References

[1] Z.-D. Bai, H.-K. Hwang and T.-H. Tsai (2003). Berry-Esseen bounds for the number of maxima in planar regions. *Electronic Journal of Probability* **8** Article No. 9, 26 pp.

[2] B. Chauvin and N. Pouyanne (2004). m-ary search trees when $m \geq 27$: a strong asymptotics for the space requirements. *Random Structures and Algorithms* **24** 133–154.

[3] H.-H. Chern and H.-K. Hwang (2001). Phase changes in random m-ary search trees and generalized quicksort. *Random Structures and Algorithms* **19** 316–358.

[4] H.-H. Chern and H.-K. Hwang (2003). Partial match queries in random quadtrees. *SIAM Journal on Computing* **32** 904–915.

[5] H.-H. Chern and H.-K. Hwang (2003). Partial match queries in random k-d trees. Manuscript submitted for publication.

[6] H.-H. Chern and H.-K. Hwang (2004). Limit distribution of the number of consecutive records. *Random Structures and Algorithms,* accepted for publication.

[7] H.-H. Chern, H.-K. Hwang and T.-H. Tsai (2003). An asymptotic theory for Cauchy-Euler differential equations with applications to the analysis of algorithms. *Journal of Algorithms* **44** 177–225.

[8] D. S. Dean and S. N. Majumdar (2002). Phase transition in a random fragmentation problem with applications to computer science. *Journal of Physics. A. Mathematical and General* **35** L501–L507.

[9] M. de Berg, M. van Kreveld, M. Overmars, O. Schwarzkopf (2000). *Computational Geometry: Algorithms and Applications*. Second revised edition, Springer-Verlag.

[10] L. Devroye (1987). Branching processes in the analysis of the heights of trees. *Acta Informatica* **24** 277–298.

[11] L. Devroye (1991). Limit laws for local counters in random binary search trees. *Random Structures and Algorithms* **2** 303–315.

[12] L. Devroye (1998). Universal limit laws for depths in random trees. *SIAM Journal on Computing* **28** 409–432.

[13] L. Devroye and L. Laforest (1990). An analysis of random d-dimensional quad trees. *SIAM Journal on Computing* **19** 821–832.

[14] A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi (1953). *Higher Transcendental Functions*. Volume I. Robert E. Krieger Publishing Co., Fla.

[15] J. A. Fill and N. Kapur (2004). The space requirement of m-ary search trees: distributional asymptotics for $m \geq 27$. In *Proceedings of the 7th Iranian Statistical Conference*.

[16] R. A. Finkel and J. L. Bentley (1974). Quadtrees: A data structure for retrieval on composite keys. *Acta Informatica* **4** 1–9.

[17] P. Flajolet, G. Gonnet, C. Puech and J. M. Robson (1993). Analytic variations on quadtrees. *Algorithmica* **10**, 473–500.

[18] P. Flajolet, X. Gourdon and C. Martínez (1997). Patterns in random binary search trees. *Random Structures and Algorithms* **11** 223–244.

[19] P. Flajolet, G. Labelle, L. Laforest, and B. Salvy (1995). Hypergeometrics and the cost structure of quadtrees. *Random Structures and Algorithms* **7** 117-114.

[20] P. Flajolet and T. Lafforgue (1994). Search costs in quadtrees and singularity perturbation asymptotics. *Discrete and Computational Geometry* **12** 151–175.

[21] P. Flajolet and A. M. Odlyzko (1990). Singularity analysis of generating functions. *SIAM Journal on Discrete Mathematics* **3** 216–240.

[22] P. Flajolet and P. Puech (1986). Partial match retrieval of multi-dimensional data. *Journal of the ACM* **32** 371–407.

[23] P. Flajolet and R. Sedgewick (1995). Mellin transforms and asymptotics: finite differences and Rice's integrals. *Theoretical Computer Science* **144**, 101–124.

[24] P. Flajolet and R. Sedgewick (1995). *Analytic Combinatorics*. Book in preparation; current version available via the link algo.inria.fr/flajolet/Publications/AnaCombi1to9.pdf.

[25] D. Foata, G.-N. Han and B. Lass (2001). Les nombres hyperharmoniques et la fratrie du collectionneur de vignettes. *Seminaire Lotharingien de Combinatoire* **47** Article B47a.

[26] M. Hoshi and P. Flajolet (1992). Page usage in a quadtree index. *BIT* **32** 384–402.

[27] H.-K. Hwang (1998). On convergence rates in the central limit theorems for combinatorial structures. *European Journal of Combinatorics,* **19** 329–343.

[28] H.-K. Hwang (2003). Second phase changes in random m-ary search trees and generalized quicksort: convergence rates. *Annals of Probability* **31** 609–629.

[29] H.-K. Hwang (2004). Phase changes in random recursive structures and algorithms (a brief survey). In *Proceedings of the Workshop on Probability with Applications to Finance and Insurance*. Edited by T. L. Lai, H. Yang and S. P. Yung, World Scientific, pp. 82–97.

[30] H.-K. Hwang and R. Neininger (2002). Phase change of limit laws in the quicksort recurrence under varying toll functions. *SIAM Journal on Computing* **31** 1687–1722.

[31] H.-K. Hwang and T.-H. Tsai (2003). An asymptotic theory for recurrence relations based on minimization and maximization. *Theoretical Computer Science* **290** 1475–1501.

[32] E. L. Ince (1926). *Ordinary Differential Equations,* Dover, New York.

[33] S. Janson (2004). Functional limit theorems for multitype branching processes and generalized Polya urns. *Stochastic Processes and their Applications* **110** 177–245.

[34] G. Labelle and L. Laforest (1995). Combinatorial variations on multidimensional quadtrees. *Journal of Combinatorial Theory, Series A* **69** 1–16.

[35] G. Labelle and L. Laforest (1995). Sur la distribution de l'arité de la racine d'une arborescence hyperquaternaire à d dimensions. *Discrete Mathematics* **139** 287–302.

[36] G. Labelle and L. Laforest (1996). Étude de constantes universelles pour les arborescences hyper-quaternaires de recherche. *Discrete Mathematics* **153** 199–211.

[37] H. M. Mahmoud (1992). *Evolution of Random Search Trees*. John Wiley & Sons, New York.

[38] C. Martínez, A. Panholzer and H. Prodinger (2001). Partial match queries in relaxed multidimensional search trees. *Algorithmica* **29** 181–204.

[39] H. N. Minh and G. Jacob (2000). Symbolic integration of meromorphic differential systems via Dirichlet functions. *Discrete Mathematics* **210** 87–116.

[40] R. Neininger and L. Rüschendorf (1999). On the internal path length of d-dimensional quad trees. *Random Structures and Algorithms* **15** 25–41.

[41] R. Neininger and L. Rüschendorf (2001). Limit laws for partial match queries in quadtrees. *Annals of Applied Probability* **11** 452–469.

[42] V. V. Petrov (1975). *Sums of Independent Random Variables*. Springer, New York.

[43] H. Samet (1990a). *The Design and Analysis of Spatial Data Structures*. Addison-Wesley, Reading, MA.

[44] H. Samet (1990b). *Applications of Spatial Data Structures: Computer Graphics, Image Processing, and GIS*. Addison-Wesley, Reading, MA.

[45] R. Sedgewick (1980). *Quicksort*. Garland, New York.