Limit Laws for Patterns in Ranked Tree-Child Networks

(joint with H. Liu and T.-C. Yu)

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September 6th, 2022

Phylogenetic Trees

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Theorem (Schröder; 1870)

We have,

$$\mathrm{T}_n = (2n-3)!!.$$

Thus, as $n \to \infty$,

$$\mathrm{T}_n \sim \frac{1}{\sqrt{2}} \left(\frac{2}{e}\right)^n n^{n-1}.$$

Patterns in Phylogenetic Trees (i)

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Theorem

Expected value and variance of the number X_n of occurrences of P are both linear. Moreover,

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathrm{Var}(X_n)}} \xrightarrow{d} N(0,1).$$

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Theorem (Chang and F.; 2010)

(i) *As* $\mathbb{E}(X_{n,k}) \to \infty$,

$$\sup_{-\infty<x<\infty} \left| P\left(\frac{X_{n,k} - \mathbb{E}(X_{n,k})}{\sqrt{\mathrm{Var}(X_{n,k})}} \le x \right) - \Phi(x) \right| = \mathcal{O}\left(\frac{1}{\sqrt{\mathrm{Var}(X_{n,k})}} \right).$$

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(ii) *As* $k \to \infty$,

$$d_{\mathrm{TV}}\left(X_{n,k}, \mathrm{Poisson}(\mathbb{E}(X_{n,k}))\right) \longrightarrow 0, \qquad (n \to \infty).$$

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H. Chang and M. Fuchs (2010). Limit theorems for patterns in phylogenetic trees, J. Math. Biol., 60:4, 481–512.

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Phylogenetic networks have become increasingly popular in recent decades.

They are used to model reticulate evolution which contains reticulation events caused by, e.g., lateral gene transfer or hybridization.

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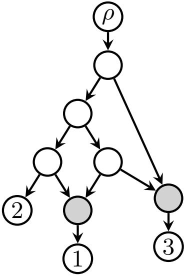
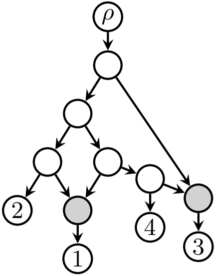


(b)



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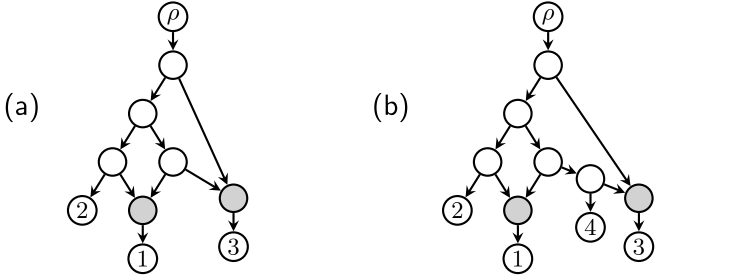

Figure: (a) is not a tc-network whereas (b) is a tc-network.

Enumeration and Pattern Counting in TC-Networks

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Theorem (F., Yu, Zhang; 2021)

We have,

$$\mathrm{TC}_n = \Theta\left(n^{-2/3} e^{a_1 (3n)^{1/3}} \left(\frac{12}{e^2}\right)^n n^{2n}\right),$$

where a_1 is the largest root of the Airy function of first order.

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The number of cherries is $o(n)$ for almost all tc-networks.

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Theorem (Chang, F., Liu, Wallner, Yu; 2023+)

We have,

$$\mathbb{E}(\#\ \textit{of cherries}) = \mathcal{O}(1).$$

Ranked TC-Networks (i)

F. Bienvenu, A. Lambert, M. Steel (2022). Combinatorial and stochastic properties of ranked tree-child networks, Random Struc. Algor., 60:4, 653–689.

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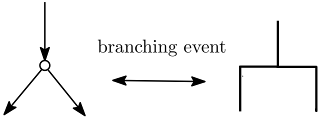

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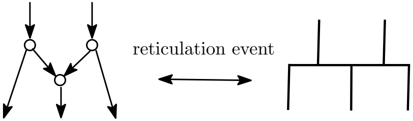

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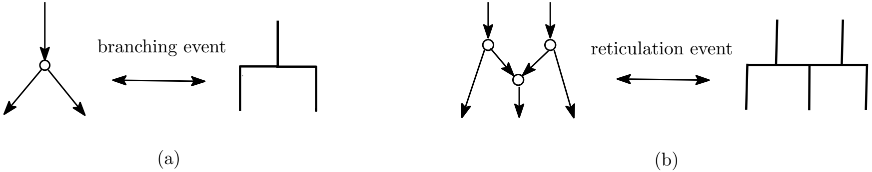
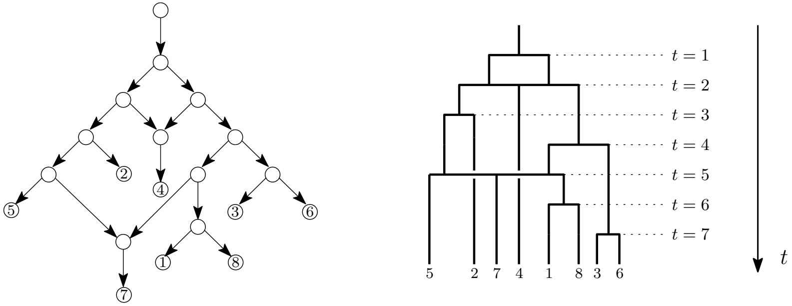


Definition

A ranked tc-network is a tc-network which is drawn starting with a branching event and consecutively adding either a branching event or a reticulation event until all events are used.

Ranked TC-Networks (ii)



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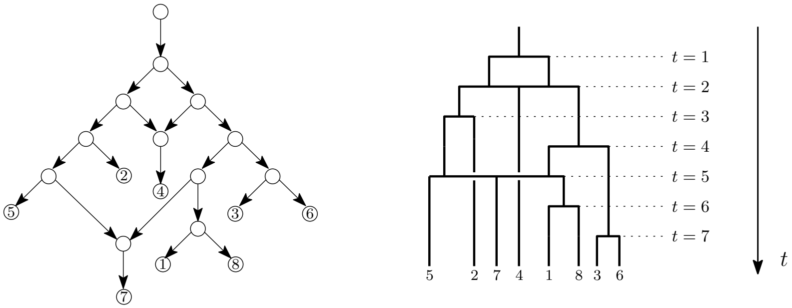

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Question: which tc-networks are rankable?

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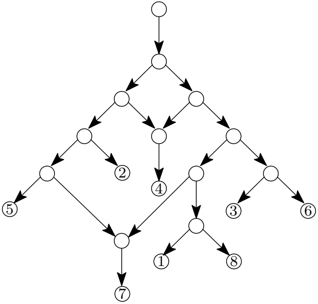

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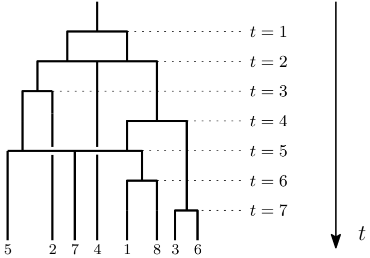

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Theorem (Bienvenu, Lambert, Steel; 2022)

The number of rankable tc-networks with n *leaves is* $o(\mathrm{TC}_n)$.

Counting Ranked TC-Networks (i)

$\mathrm{RTC}_{n,k}$ ... # of ranked tc-networks with k reticulation nodes.

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Theorem (Bienvenu, Lambert, Steel; 2022)

We have,

$$\mathrm{RTC}_{n,k} = \left[{n-1 \atop n-1-k} \right] \cdot \frac{n!(n-1)!}{2^{n-1}},$$

where $\left[{n-1 \atop n-1-k} \right]$ *denotes the unsigned Stirling numbers of first kind and* $n!(n-1)!/2^{n-1}$ *is the number of ranked trees.*

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Corollary

We have,

$$\frac{\#\ \textit{of reticulation nodes} - n + \log n}{\sqrt{\log n}} \xrightarrow{d} N(0,1).$$

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We have,

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This is $A167484$ in the OEIS (www.oeis.org):

> *Assume that n people are on one side of a river. Then, this sequences is the number of ways to cross to the other side with a two-person boat if the crossings follow the pattern 2 sent, 1 returns, 2 sent, 1 returns, ..., 2 sent.*

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Caraceni, F., Yu (2022) found a natural bijection $\longrightarrow$ Guan-Ru Yu's talk.

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$X_n \ldots$ # of occurrences of a pattern in the resulting tree.

Lemma

X_n has the same distribution as the number of occurrences of the pattern in a random ranked tc-network with n leaves.

Cherries and Tridents

$C_n \ldots$ # of cherries of a random ranked tc-network of size n;

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Theorem (Bienvenu, Lambert, Steel; 2022)

- *We have,* $C_n \xrightarrow{d} \mathrm{Poisson}(1/4)$.
- *We have,* $T_n/n \xrightarrow{\mathbb{P}} 1/7$.

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We have $T_2 = 1$ and

$$(T_{n+1}|T_n = j) = \begin{cases} j-1, & \text{with probability } 3j(3j-2)/n^2; \\ j+1, & \text{with probability } (n-3j)(n-3j-1)/n^2; \\ j, & \text{otherwise.} \end{cases}$$

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We have,

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Set

$$\phi_{n,m} := \mathbb{E}(T_n - \mu_n)^m,$$

i.e., $\phi_{n,m}$ is the m-th central moment of T_n.

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The m-th central moment satisfies:

$$\phi_{n+1} = \left(1 - \frac{\kappa}{n}\right)^2 \phi_n + \psi_n,$$

with $\kappa = 3m$ and ψ_n depends on k-th central moments with $k < m$.

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Proposition

For $m \geq 2$,

$$\mathbb{E}(T_n - \mu_n)^m \sim \mathbb{E}(N(0,1)^m) \left(\frac{24}{637}\right)^{m/2} n^{m/2}.$$

CLT for Tridents (iii)

Theorem

Assume that $\mathbb{E}(X_n^k) \longrightarrow m_k$ *for all* $k \geq 1$.

Then, there exists a distribution X *with* $\mathbb{E}(X^k) = m_k$.

Moreover, if X *is uniquely characterised by its sequence of moments, then*

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Theorem (F., Liu, Yu; 2023)

We have,

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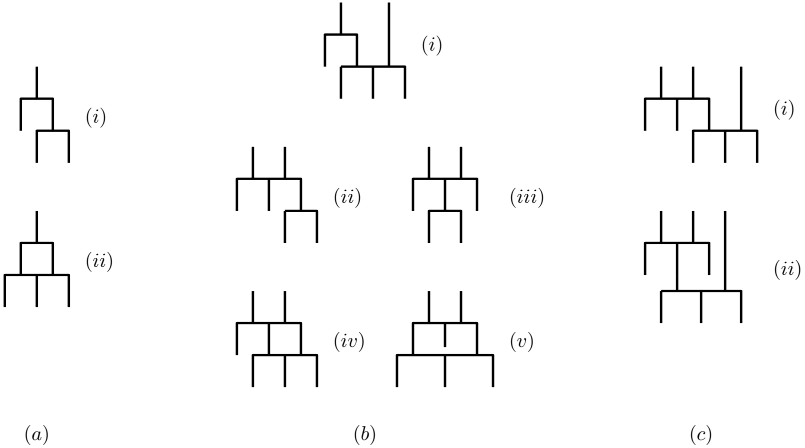
Patterns of Height 2



Limit Laws for Patterns of Height 2

Theorem (F., Liu, Yu; 2023)

(a) *The patterns in (a) have a degenerate limit law. More precisely,*

$$X_n \xrightarrow{L_1} 0.$$

(b) *For the patterns in (b), we have*

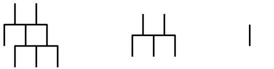
$$X_n \xrightarrow{d} \mathrm{Poisson}(\lambda),$$

where $\lambda = 1/8$ *or* $1/28$ *or* $1/56$ *or* $1/14$ *or* $1/28$.

(c) *For the patterns in (c), we have*

$$\frac{X_n - \mu n}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1).$$

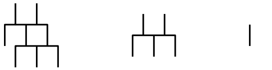
Pattern (b-iv)



type A type B type C

(a) (b) (c)

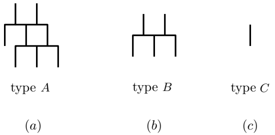
Pattern (b-iv)



type A (a) | type B (b) | type C (c)

	type A	type B	probability
A	-1	0	$4a/n^2$
B	0	-1	$3b/n^2$
C	0	0	c/n^2

Pattern (b-iv)



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C	0	0	c/n^2

	type A	type B	probability
A	-1	$+1$	$8a/n^2$
	0	0	$4a/n^2$
A & A	-2	$+1$	$9a(a-1)/n^2$
	-2	$+2$	$6a(a-1)/n^2$
	-2	$+3$	$a(a-1)/n^2$
B	0	0	$2b/n^2$
	$+1$	-1	$4b/n^2$
B & B	0	-1	$9b(b-1)/n^2$
C & C	0	$+1$	$c(c-1)/n^2$
A & B	-1	0	$18ab/n^2$
	-1	$+1$	$6ab/n^2$
A & C	-1	$+1$	$6ac/n^2$
	-1	$+2$	$2ac/n^2$
B & C	0	0	$6bc/n^2$

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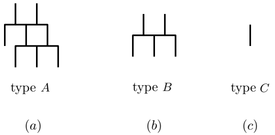

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A	-1	0	$4a/n^2$
B	0	-1	$3b/n^2$
C	0	0	c/n^2

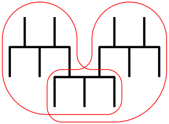
	type A	type B	probability
A	-1	$+1$	$8a/n^2$
	0	0	$4a/n^2$
A & A	-2	$+1$	$9a(a-1)/n^2$
	-2	$+2$	$6a(a-1)/n^2$
	-2	$+3$	$a(a-1)/n^2$
B	0	0	$2b/n^2$
	$+1$	-1	$4b/n^2$
B & B	0	-1	$9b(b-1)/n^2$
C & C	0	$+1$	$c(c-1)/n^2$
A & B	-1	0	$18ab/n^2$
	-1	$+1$	$6ab/n^2$
A & C	-1	$+1$	$6ac/n^2$
	-1	$+2$	$2ac/n^2$
B & C	0	0	$6bc/n^2$

Proposition

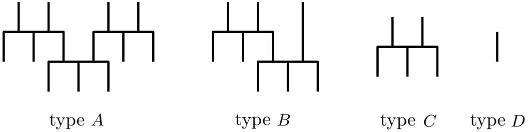
We have,

$$\mathbb{E}(X_n^r T_n^s) \sim \frac{n^s}{14^r 7^s}.$$

Pattern (c-i)

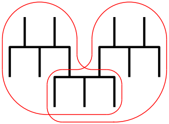


(a)



(b)

Pattern (c-i)



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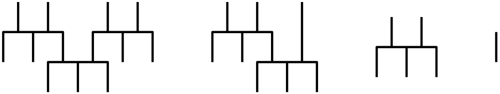

type A type B type C type D

(b)

We have,

$$\mathbb{E}(X_n) = \frac{(1080n^5-16668n^4+96992n^3-261735n^2+319471n-135654)n}{20790(n-1)(n-2)(n-3)(n-4)(n-5)}$$

and

$$\mathbb{E}(Y_n)=\frac{2(4290n^7-125730n^6+1509970n^5-9550275n^4+33968326n^3-66905671n^2+66128140n-24510098)n}{1576575(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}$$

where Y_n is the number of occurrences of (a).

Limit Law of Pattern (c-i)

Proposition

We have,

$$\mathbb{E}((Y_n - \mathbb{E}(T_n))^r (X_n - \mathbb{E}(X_n))^s (T_n - \mathbb{E}(T_n))^t) \sim \mathbb{E}(N_1^r N_2^s N_3^t) n^{(r+s+t)/2}.$$

where (N_1, N_2, N_3) *has distribution* $N(\mathbf{0}, \Sigma)$ *with*

$$\Sigma = \begin{bmatrix} \frac{1002796}{203664825} & \frac{433528}{62537475} & \frac{-32}{13377} \\ \frac{433528}{62537475} & \frac{4575916}{137582445} & -\frac{608}{119119} \\ \frac{-32}{13377} & -\frac{608}{119119} & \frac{24}{637} \end{bmatrix}.$$

Thus,

$$\frac{1}{\sqrt{n}} (Y_n - \mathbb{E}(Y_n), X_n - \mathbb{E}(X_n), T_n - \mathbb{E}(T_n)) \xrightarrow{d} N(\mathbf{0}, \Sigma).$$

Conjecture for general Patterns

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Conjecture

(a) *If P is a normal pattern, then F is a Poisson pattern; in all other cases, F is a degenerate pattern.*

(b) *If P_1, P_2 are both normal patterns, then F is a normal pattern; if P_1 is a normal pattern and P_2 is a Poisson pattern or vice versa, then F is a Poisson pattern; in all other cases, F is a degenerate pattern.*

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The proof would require a less computational-intensive approach.

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  $$u(n,k)=ku(n-1,k-1)+\left(\binom{n}{2}-\binom{2k}{2}\right)u(n-1,k)+3\binom{n-2k}{3}u(n-1,k+1),$$

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  Asymptotics of $u(n,0)$?
- How about stochastic results for random ranked galled trees?

References

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