Enumeration and Stochastic Properties of Tree-Child Networks

(based on joint work with Y.-S. Chang, H. Liu, M. Wallner, G.-R. Yu and L. Zhang)

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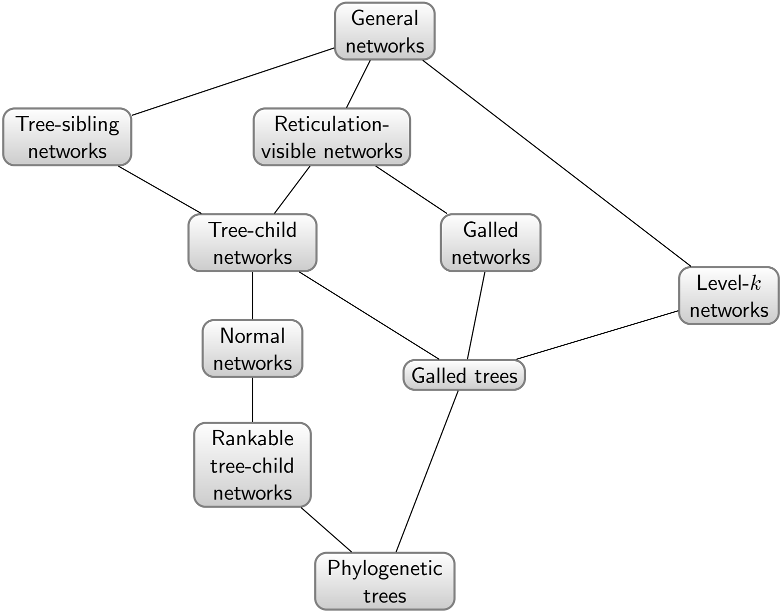
Phylogenetic networks have become increasingly popular in recent decades.

They are used to model reticulate evolution which contains reticulation events such as lateral gene transfer or hybridization.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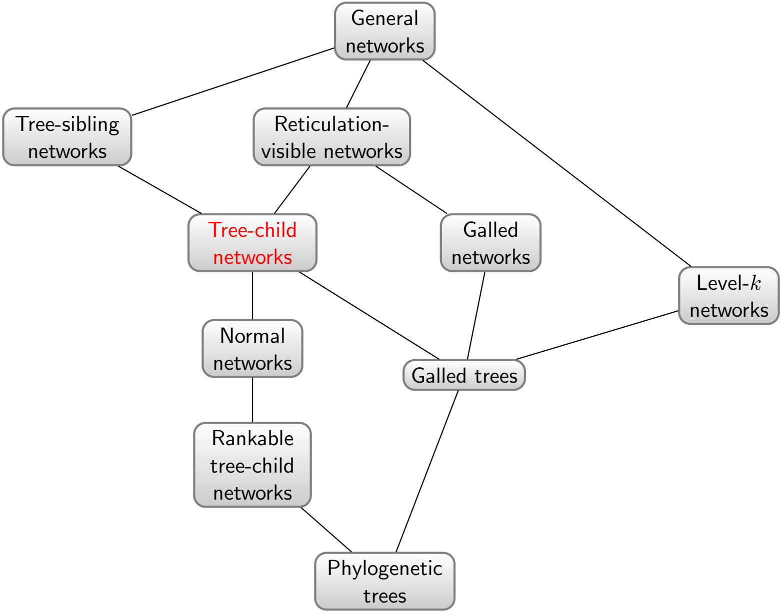
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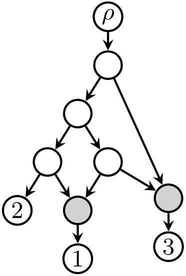

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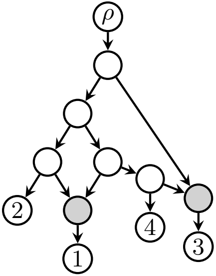

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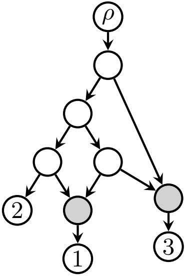

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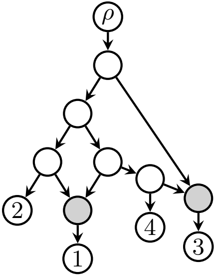

Figure: (a) is not a tc-network whereas (b) is a tc-network.

Counting TC-Networks

TC_n ... $\#$ of tc-networks with n leaves.

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Theorem (McDiarmid, Semple, Welsh; 2015)

For constants $0 < c_1 < c_2$,

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Theorem (McDiarmid, Semple, Welsh; 2015)

(a) *$\#$ of reticulation nodes $\sim n$ for almost all tc-networks;*

(b) *The number of cherries is $o(n)$ for almost all tc-networks.*

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Cardona & Zhang (2020):

$k \setminus n$	2	3	4	5	6	7
1	2	21	228	2805	39330	623385
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Computation becomes hard for large n with the method of Cardona & Zhang which uses component graphs whose number increases rapidly!

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Chang & Liu & F. & Wallner & Yu (2023+) recently also found the following recursive formula:

$$\mathrm{TC}_{n,k} = \frac{n!}{2^{n-1-k}} w_{n-1,k},$$

where

$$\omega_{n,k} = \sum_{m\geq 1} b_{n,k,m}$$

with $b_{n,km}$ given recursively by:

$$b_{n,k,m} = \sum_{j=1}^{m} b_{n-1,k,j} + (n+m+k-2) \sum_{j=1}^{m} b_{n-1,k-1,j}.$$

A Sequence of Words

Definition (OEIS; A213863)

Denote by a_n the number of words on letters $\{\omega_1, \dots, \omega_n\}$ so that

(i) *each letter occurs exactly 3 times;*

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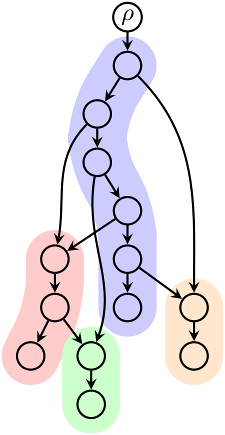
Proposition (F., Yu, Zhang; 2021)

We have,

$$\frac{\mathrm{TC}_{n,n-1}}{n!} = a_{n-1}.$$

Bijective Proof



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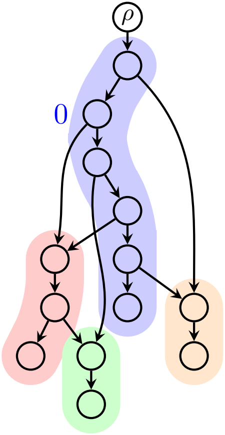

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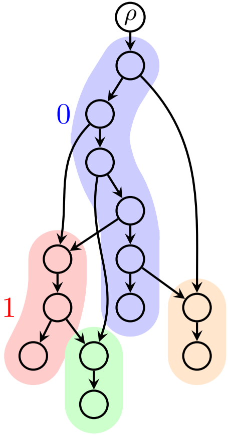

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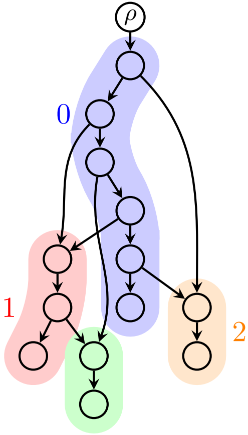

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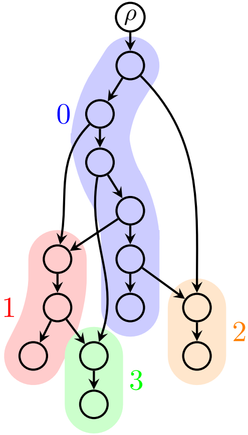

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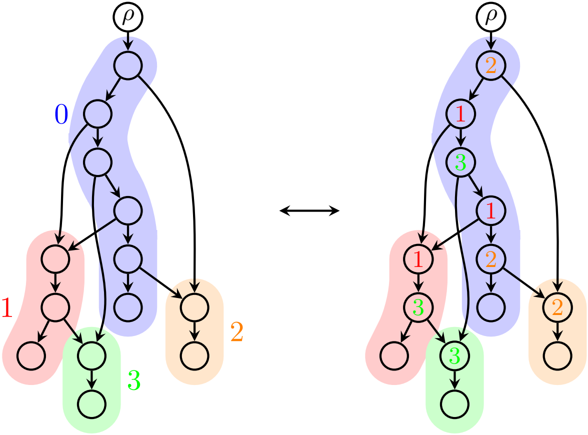

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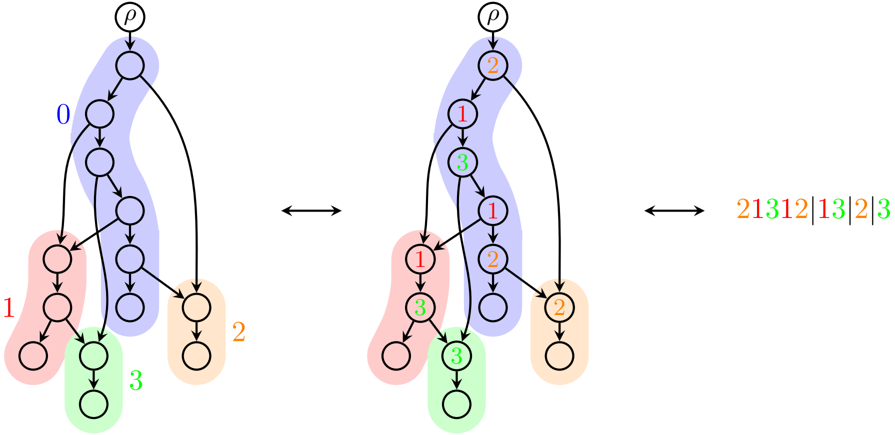

Asymptotic Counting of TC-Networks

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Theorem (F., Yu, Zhang; 2021)

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$$\mathrm{TC}_n = \Theta(\mathrm{TC}_{n,n-1}) = \Theta\left(n^{-2/3}e^{a_1(3n)^{1/3}}\left(\frac{12}{e^2}\right)^n n^{2n}\right),$$

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So, the base of the exponential growth rate is $12/e^2$!

Stochastic Results for TC-Networks

Chang & F. & Liu & Wallner & Yu (2023+) proved that

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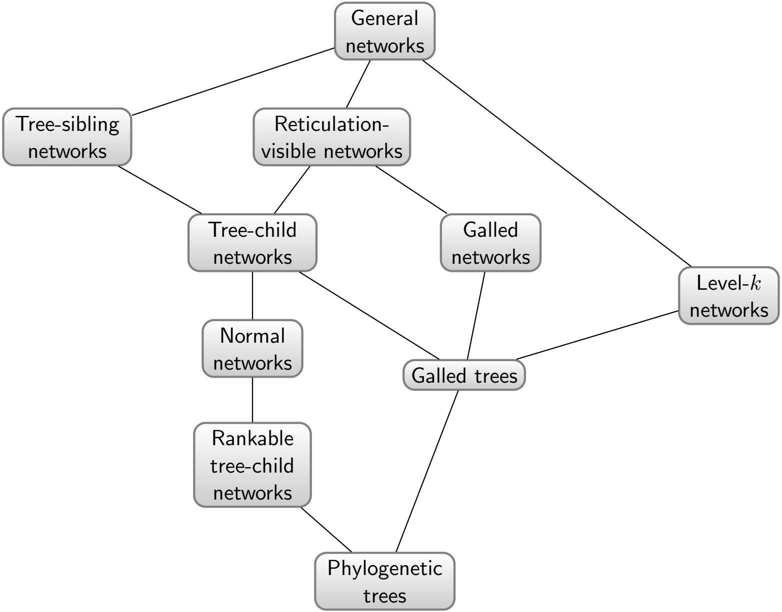
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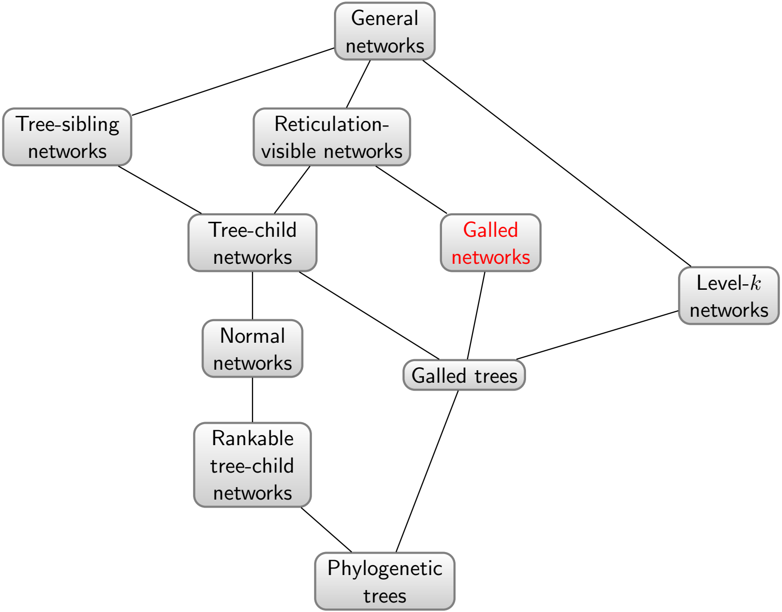
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F. & Yu & Zhang (2022) also found the limiting distribution of the number X_n of reticulation nodes. In particular,

$$\mathbb{E}(X_n) = n - \frac{3}{8} + o(1) \qquad \text{and} \qquad \mathrm{Var}(X_n) = \frac{3}{4} + o(1).$$

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- Enumeration and stochastic results for reticulation-visible networks?

References

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