Limit Laws for the Number of Groups formed by Social Animals under the Extra Clustering Model

(joint with Michael Drmota and Yi-Wen Lee)

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Probabilistic Analysis of a Genealogical Model of Animal Group Patterns

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Mathematical Biology

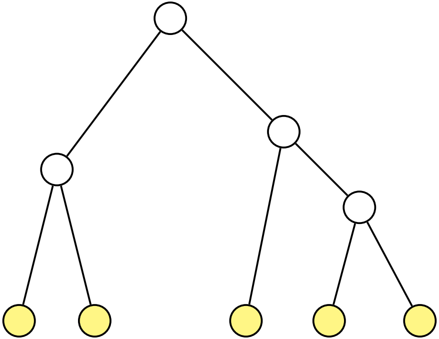
Probabilistic analysis of a genealogical model of animal group patterns

Eric Durand · Olivier François

Received: 26 October 2007 / Revised: 5 March 2009 / Published online: 12 April 2009

Phylogenetic Tree

Ordered, binary, rooted tree with leafs representing the animals.



Describes the genetic relatedness of animals.

Yule-Harding Model (Bottom-Up)

Fundamental random model in phylogenetics.

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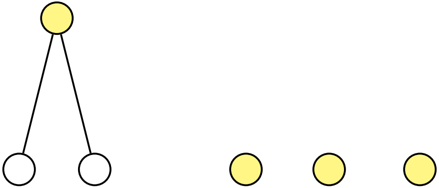

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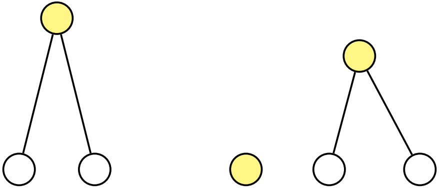

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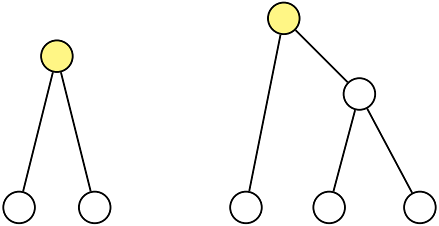

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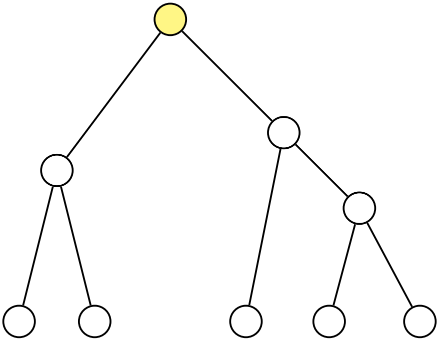

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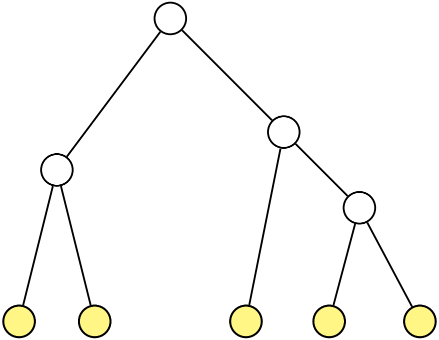

Animal Groups under the Yule-Harding Model

Durand, Blum and François (2007):

Groups are formed more likely by animals which are genetically related.

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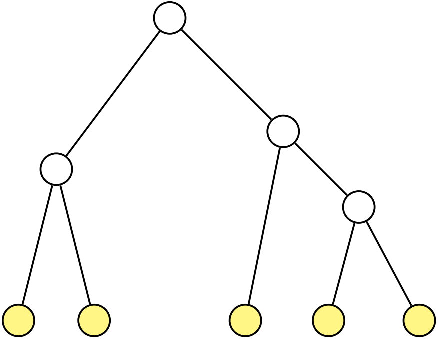

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All leafs of the tree rooted at the parent.

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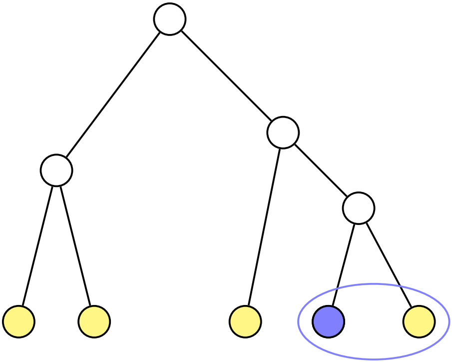

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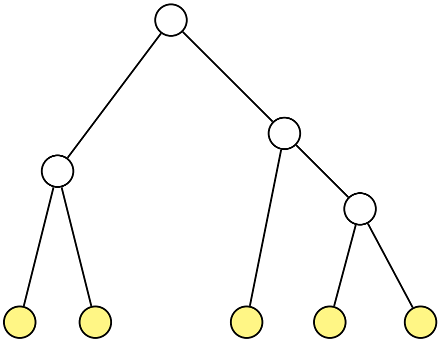

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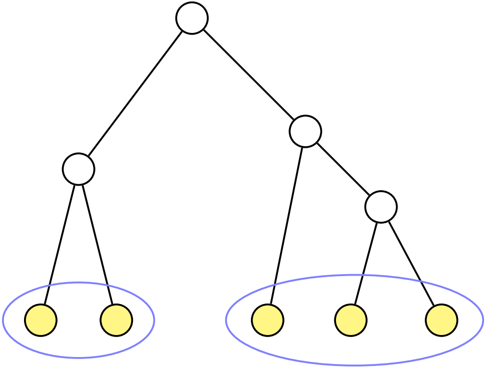

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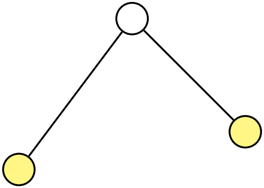

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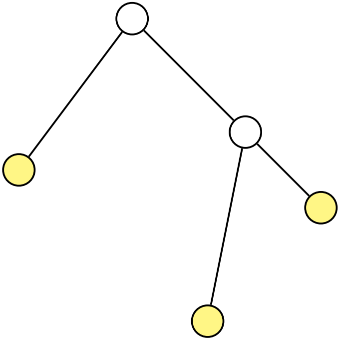

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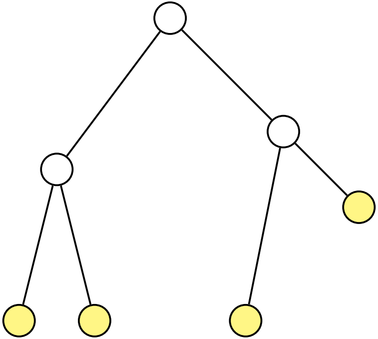

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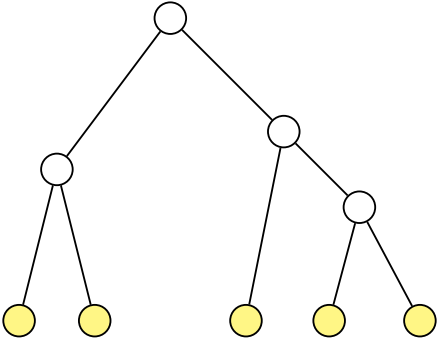

of Groups

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$$X_n \stackrel{d}{=} \begin{cases} 1, & \text{if } I_n = 1 \text{ or } I_n = n-1, \\ X_{I_n} + X^*_{n-I_n}, & \text{otherwise,} \end{cases}$$

where $I_n = \text{Uniform}\{1, \ldots, n-1\}$ is the # of animals in the left subtree and X_n^* is an independent copy of X_n.

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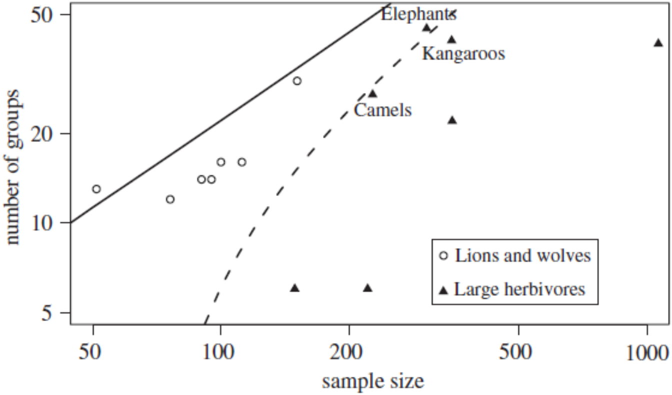
Theorem (Durand and François; 2010)

We have,

$$\mathbb{E}(X_n) \sim an \qquad \left(a := \frac{1-e^{-2}}{4}\right).$$

Comparison with Real-life Data

Durand, Blum and François (2007) presented the following data:



Extra Clustering Model

Durand, Blum and François (2007):

Let $p \geq 0$.

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Introduced to test whether or not genetic relatedness is the sole driving force behind the group formation process.

Average Number of Groups

Theorem (Durand and François; 2010)

We have,

$$\mathbb{E}(X_n) \sim \begin{cases} \dfrac{c(p)}{\Gamma(2(1-p))} n^{1-2p}, & \textit{if } p < 1/2; \\ \dfrac{\log n}{2}, & \textit{if } p = 1/2; \\ \dfrac{p}{2p-1}, & \textit{if } p > 1/2, \end{cases}$$

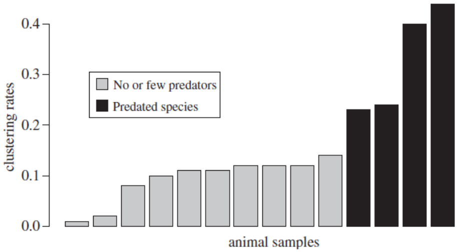
where

$$c(p) := \frac{1}{e^{2(1-p)}} \int_0^1 (1-t)^{-2p} e^{2(1-p)t} \left(1-(1-p)t^2\right) \mathrm{d}t.$$

Testing for the Neutral Model

Durand, Blum and François (2007):

	Sample size	Number of herds	Rate $\hat{p}$
(A)			
Springboks (browsers)	149	6	0.40
Springboks (graze)	1064	40	0.24
Fallow deers	349	22	0.23
Grant's gazelles	221	6	0.44
Wild camels	227	27	0.14
Kangaroos	348	41	0.12
African savannah elephants	304	45	0.08
	Sample size	**Number of packs/prides**	**Rate $\hat{p}$**
(B)			
Yellowstone Wolves 2002	90	14	0.11
Yellowstone Wolves 2004	112	16	0.12
Alaska Wolves	151	30	0.02
Scandinavian wolf	76	12	0.11
Zambia Kafue lions	95	14	0.12
Selous Game lions	51	13	0.00
Serengeti lions	100	16	0.10



Yi-Wen's Thesis (2012)

Group Patterns of Social Animals under the Neutral Model

Yi-Wen Lee

Department of Applied Mathematics,

National Chiao Tung University

This thesis was supervised by Dr. Michael Fuchs

May 26, 2012

Variance and SLLN

Theorem (Lee; 2012)

We have,

$$\mathrm{Var}(X_n) \sim \frac{(1-e^{-2})^2}{4} n \log n = 4a^2 n \log n.$$

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We have,

$$P\left(\lim_{n\to\infty}\left|\frac{X_n}{\mathbb{E}(X_n)}-1\right|=0\right)=1.$$

For SLLN, X_n is constructed on the same probability space via the tree evolution process underlying the Yule-Harding model.

Higher Moments

Theorem (Lee; 2012)

For all $k \geq 3$,

$$\mathbb{E}(X_n - \mathbb{E}(X_n))^k \sim (-1)^k \frac{2k}{k-2} a^k n^{k-1}.$$

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$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathrm{Var}(X_n)}}$$

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Question: Is there a limit distribution?

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Unordered, rooted trees.

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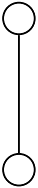

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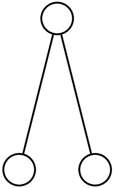

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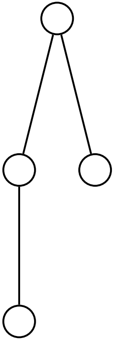

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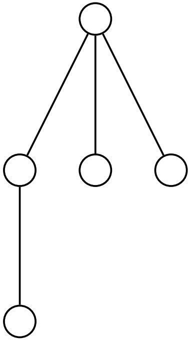

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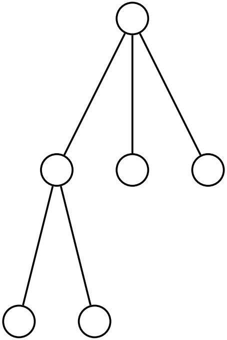

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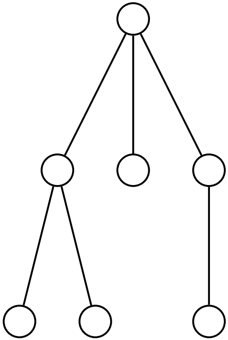

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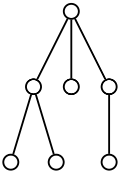
Meir and Moon (1974):

Randomly pick an edge and remove it; retain the tree containing the root.



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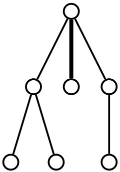

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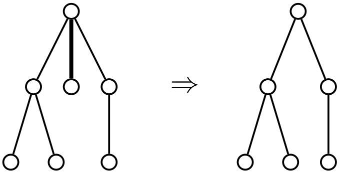

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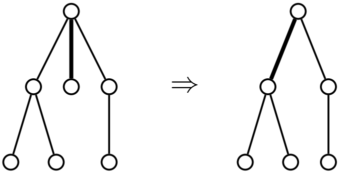

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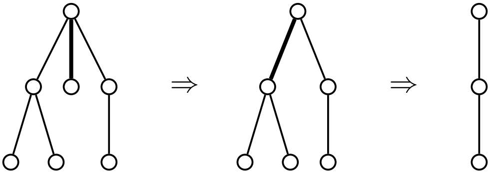

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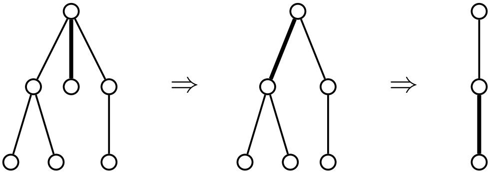

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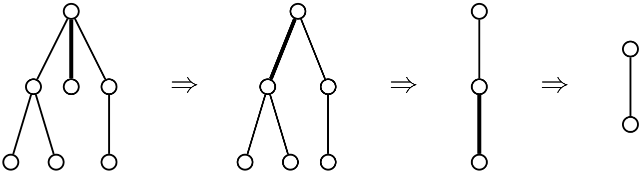

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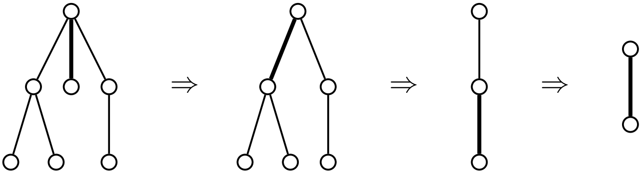

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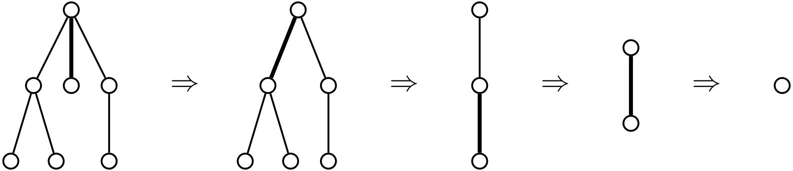

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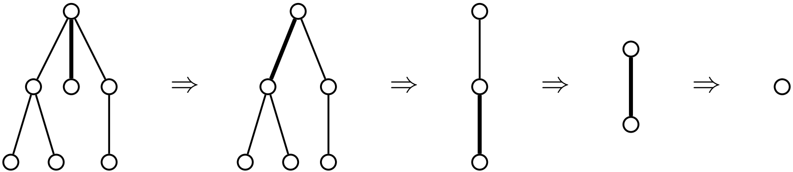

$$Y_n = \text{ number of steps until tree is destroyed}$$
$$= \text{ number of edges cut} = 4.$$

Mean, Variance and Higher Moments

Theorem (Panholzer; 2004)

We have,

$$\mathbb{E}(Y_n) \sim \frac{n}{\log n}$$

and for $k \geq 2$

$$\mathbb{E}(Y_n - \mathbb{E}(Y_n))^k \sim \frac{(-1)^k}{k(k-1)} \cdot \frac{n^k}{\log^{k+1} n}.$$

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$$\mathbb{E}(Y_n) \sim \frac{n}{\log n}$$

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$$\mathbb{E}(Y_n - \mathbb{E}(Y_n))^k \sim \frac{(-1)^k}{k(k-1)} \cdot \frac{n^k}{\log^{k+1} n}.$$

Thus, again the limit law of

$$\frac{Y_n - \mathbb{E}(Y_n)}{\sqrt{\mathrm{Var}(Y_n)}}$$

cannot obtained from the method of moments!

Limit Law

Theorem (Drmota, Iksanov, Moehle, Roessler; 2009)

We have,

$$\frac{\log^2 n}{n} Y_n - \log n - \log\log n \xrightarrow{d} Y$$

with

$$\mathbb{E}(e^{i\lambda Y}) = e^{i\lambda \log|\lambda| - \pi|\lambda|/2}.$$

The law of Y *is spectrally negative stable with index of stability* 1.

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The law of Y is spectrally negative stable with index of stability 1.

Different proofs of this result exist.

Limit Law of X_n

Recall that

$$X_n \stackrel{d}{=} \begin{cases} 1, & \text{if } I_n = 1 \text{ or } I_n = n-1, \\ X_{I_n} + X^*_{n-I_n}, & \text{otherwise,} \end{cases}$$

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Theorem (Drmota, F., Lee; 2014)

We have,

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\text{Var}(X_n)/2}} \xrightarrow{d} N(0,1).$$

Some Ideas of the Proof (i)

Set

$$X(y,z)=\sum_{n\ge 2}\mathbb{E}\left(e^{yX_n}\right)z^n.$$

Then,

$$z\frac{\partial}{\partial z}X(y,z)=X(y,z)+X^2(y,z)+e^y z^2\frac{2e^y z^3}{1-z}.$$

This is a Riccati DE.

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This is a Riccati DE.

Set

$$\tilde{X}(y,z)=\frac{X(y,z)}{z}.$$

Then,

$$\frac{\partial}{\partial z}\tilde{X}(y,z)=\tilde{X}^2(y,z)+e^y\frac{1+z}{1-z}.$$

Some Ideas of the Proof (ii)

Set

$$\tilde{X}(y,z) = -\frac{V'(y,z)}{V(y,z)}.$$

Then,

$$V''(y,z) + e^y \frac{1+z}{1-z} V(y,z) = 0.$$

This is Whittaker's DE.

Some Ideas of the Proof (ii)

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Solution is given by

$$V(y,z) = M_{-e^{y/2},1/2}\left(2e^{y/2}(z-1)\right) + c(y) W_{-e^{y/2},1/2}\left(2e^{y/2}(z-1)\right),$$

where

$$c(y) = -\frac{\left(e^{y/2}-1\right) M_{-e^{y/2}+1,1/2}\left(-2e^{y/2}\right)}{W_{-e^{y/2}+1,1/2}\left(-2e^{y/2}\right)}.$$

Some Ideas of the Proof (iii)

Lemma

$V(y,z)$ *is analytic in*

$$\Delta = \{z \in \mathbb{C} \ : \ |z| < 1+\delta, \arg(z) \neq \pi\}$$

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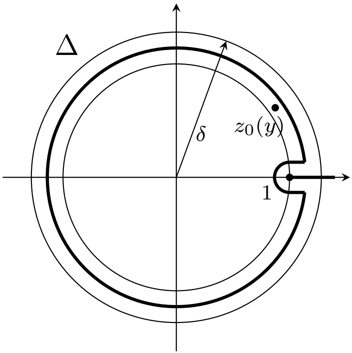

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Let $y = it/(2a\sqrt{n \log n})$. Then,

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This together with the expansion of $z_0(y)$ yields

$$\mathbb{E}\left(e^{yX_n}\right) = \exp\left(\frac{it\sqrt{n}}{2\sqrt{\log n}} - \frac{t^2}{4}\right)\left(1 + \mathcal{O}\left(\frac{\log\log n}{\log n}\right)\right).$$

Extra Clustering Model: $0 < p < 1/2$ (i)

Theorem (Drmota, F., Lee; 2014)

We have,

$$\frac{X_n}{n^{1-2p}} \xrightarrow{d} X,$$

where the distribution of X is the sum of a discrete distribution with mass $p/(1-p)$ at 0 and a continuous distribution on $[0,\infty)$ with density

$$f(x) = \frac{4(1-2p)^3}{1-p} \sum_{k \geq 0} \frac{(-\delta(p))^k}{k!\Gamma(2(k+1)p-k)} x^k,$$

where

$$\delta(p) = \frac{(1-2p)^2 W_{p,(1-2p)/p}(-2(1-p))}{4^{p-1}(1-p)^{2p} M_{p,(1-2p)/p}(-2(1-p))}.$$

Extra Clustering Model: $0 < p < 1/2$ (ii)

We have $\mathbb{E}(X^k) = d_k/\Gamma(k(1-2p)+1)$ with

$$d_1 = \frac{1}{e^{2(1-p)}} \int_0^1 (1-t)^{-2p} e^{2(1-p)t} \left(1 - (1-p)t^2\right) \mathrm{d}t$$

and for $k \geq 2$

$$d_k = \frac{2(1-p)}{(k-1)(1-2p)} \sum_{j=0}^{k-2} \binom{k-1}{j} d_{k-1-j} d_{j+1}.$$

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Moreover,

$$\mathbb{E}\left(e^{yX}\right) = \frac{1}{2\pi i} \int_{\mathcal{H}} \Phi(y,t) e^{-t} \mathrm{d}t,$$

where $\mathcal{H}$ is the Hankel contour and

$$\Phi(y,t) = \frac{4(1-2p)^2 - ypm(p)4^p(1-p)^{2p-1}t^{2p-1}}{4(1-2p)^2 t - ym(p)4^p(1-p)^{2p}t^{2p}}$$

and $m(p) = M_{p,(1-2p)/2}(-2(1-p))/W_{p,(1-2p)/2}(-2(1-p))$.

Extra Clustering Model: $p = 1/2$

Theorem (Drmota, F., Lee; 2014)

We have,

$$\mathbb{E}(X_n^k) \sim \frac{k! J_{2k-1}}{(2k-1)! 2^{2k-1}} \log^{2k-1} n,$$

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For the moments, we have $\mathbb{E}(X^k) = e_k$ with $e_1 = p/(2p-1)$ and for $k \geq 2$

$$e_k = \frac{2(1-p)}{2p-1} \sum_{j=0}^{k-2} \binom{k-1}{j} e_{k-1-j} e_{j+1} + \frac{p}{2p-1}.$$

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  Any relationship?
- How about limit laws for X_n for other random tree models?