Approximate Counting via the Poisson-Laplace-Mellin Method

(joint with Chung-Kuei Lee and Helmut Prodinger)

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Hsinchu, Taiwan

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Answer: Allow an error tolerance: **approximate counting**.

Counter C_n with $C_0 = 0$ and $(0 < q < 1)$

$$C_{n+1} = \begin{cases} C_n + 1, & \text{with probability } q^{C_n}; \\ C_n, & \text{with probability } 1 - q^{C_n}. \end{cases}$$

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Easy to show:

$$\mathbb{E}(q^{-C_n}) = n\left(q^{-1} - 1\right) + 1.$$

Now, only $\Theta(\log \log n)$ space is needed.

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Many refinements have been proposed.

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- Size of greedy independent set in random graphs.
- Size of greedy clique in random graphs.
- Length of leftmost path in random digital search trees.

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Variations of this Markov chain were also studied:

Simon (1988); Crippa and Simon (1997); Bertoin, Biane and Yor (2003); Guillemin, Robert and Zwart (2004); Louchard and Prodinger (2008)

Analysis of Approximate Counting

Flajolet (1985):

$$\mathbb{E}(C_n) \sim \log_{1/q} n + C_{\text{mean}} + F(\log_{1/q} n),$$

where $F(z)$ is a 1-periodic function

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where $F(z)$ is a 1-periodic function and

$$\text{Var}(C_n) \sim C_{\text{var}} + G(\log_{1/q} n),$$

where $G(z)$ is a 1-periodic function and

$$C_{\text{var}} = \frac{\pi^2}{6\log^2(1/q)} - \alpha - \beta + \frac{1}{12} - \frac{1}{\log(1/q)} \sum_{l\geq 1} \frac{1}{l \sinh(2l\pi^2/\log(1/q))}$$

with $\alpha = \sum_{l\geq 1} q^l/(1-q^l)$ and $\beta = \sum_{l\geq 1} q^{2l}/(1-q^l)^2$.

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- **Probability Theory**: Robert (2005)

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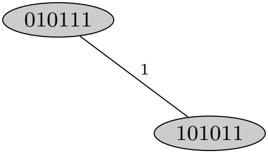

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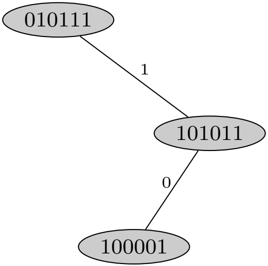

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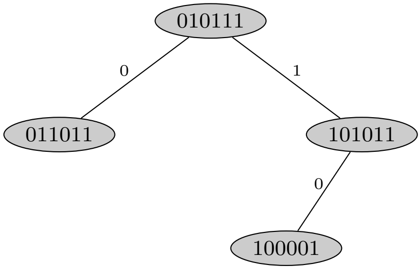

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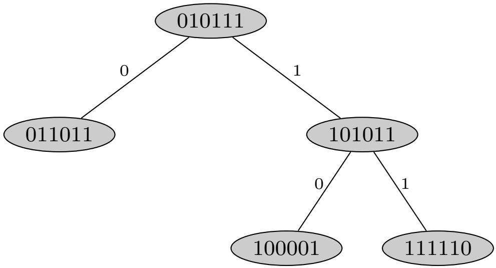

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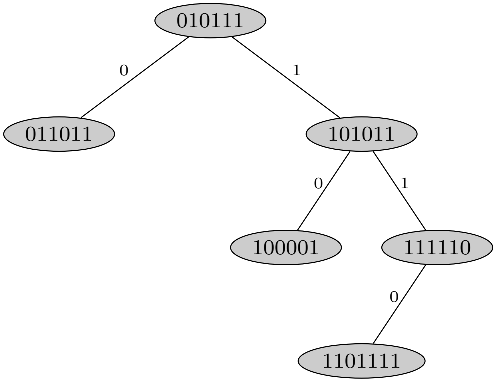

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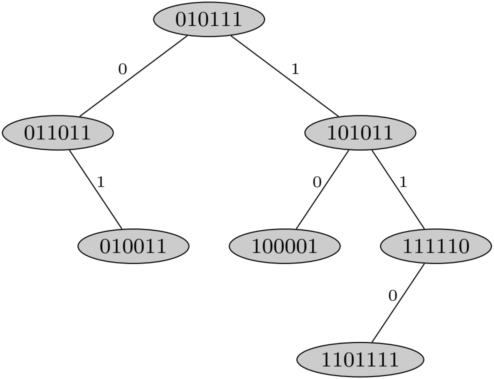

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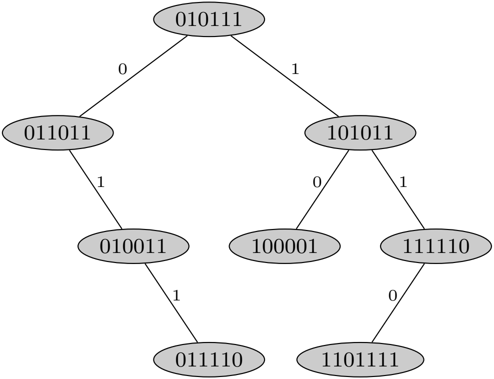

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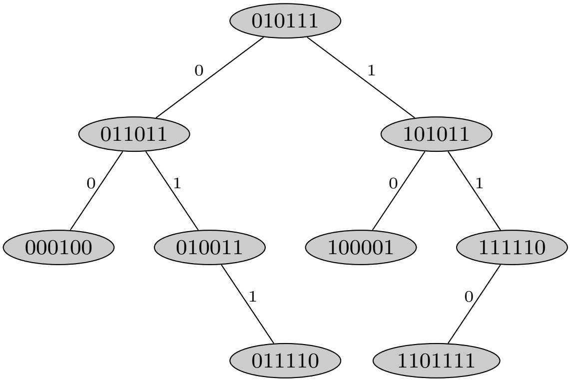

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Bits are generated by independent Bernoulli random variables with mean p.

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X_n: number of vertices on leftmost path.

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X_n: number of vertices on leftmost path.

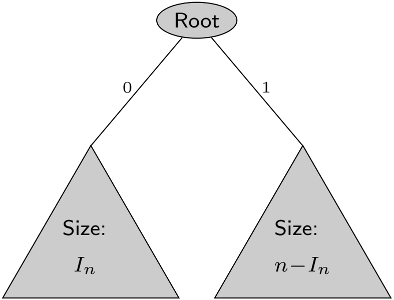
Note that:

$$X_n \stackrel{d}{=} C_n.$$

Distributional Recurrence of X_n

$$X_{n+1} \stackrel{d}{=} X_{I_n} + 1$$

- $I_n \stackrel{d}{=} \text{Binomial}(n, q)$;
- X_n, I_n independent.



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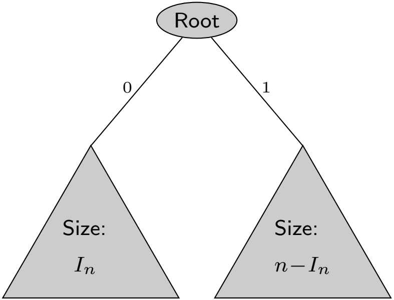

Recurrence of moments:

$$f_{n+1} = \sum_{j=0}^{n} \binom{n}{j} q^j p^{n-j} f_j + g_n.$$

Other Shape Parameters

- **Depth**

  Konheim, Newman, Knuth, Devroye, Louchard, Szpankowski

- **Total Path Length**

  Flajolet, Sedgewick, Prodinger, Kirschenhofer, Szpankowski, Hubalek

- **Peripheral Path Length**

  Drmota, Gittenberger, Panholzer, Prodinger, Ward

- **# of Occurrences of Patterns**

  Knuth, Flajolet, Sedgewick, Prodinger, Kirschenhofer

- **Colless Index**

  Fuchs, Hwang, Zacharovas

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- **Schachinger's Approach**

  Largely elementary.

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- We use inverse Mellin transform and inverse Laplace transform to obtain asymptotic expansions in the Poisson model.

Poissonization

Moments satisfy the recurrence:

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Consider Poisson-generating function of f_n and g_n, i.e.,

$$\tilde{f}(z) := e^{-z} \sum_{n\geq 0} f_n \frac{z^n}{n!}, \qquad \tilde{g}(z) := e^{-z} \sum_{n\geq 0} g_n \frac{z^n}{n!}.$$

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Then,

$$\tilde{f}(z) + \tilde{f}'(z) = \tilde{f}(qz) + \tilde{g}(z).$$

This is a differential-functional equation.

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More precisely: if f_n is smooth enough,

$$f_n \sim \sum_{j\geq 0} \frac{\tilde{f}^{(j)}(n)}{n!}\tau_j(n) = \tilde{f}(n) - \frac{n}{2}\tilde{f}''(n) + \dots,$$

where $\tau_j(n) := n![z^n](z-n)^j e^z$

This is called *Poisson-Charlier expansion* (can be already found in Ramanujan's notebooks).

Jacquet-Szpankowski-admissibility (JS-admissibility)

$\tilde{f}(z)$ is called JS-admissible if

(I) Uniformly for $|\arg(z)| \le \epsilon$,

$$\tilde{f}(z) = \mathcal{O}\left(|z|^{\alpha} \log^{\beta} |z|\right),$$

(O) Uniformly for $\epsilon < |\arg(z)| \le \pi$,

$$f(z) := e^{z}\tilde{f}(z) = \mathcal{O}\left(e^{(1-\epsilon)|z|}\right).$$

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Theorem (Jacquet and Szpankowski)

If $\tilde{f}(z)$ is JS-admissible, then

$$f_n \sim \tilde{f}(n) - \frac{n}{2}\tilde{f}''(n) + \cdots .$$

Depoissonization

JS-admissibility satisfies closure properties:

(i) $\tilde{f}, \tilde{g}$ JS-admissible, then $\tilde{f} + \tilde{g}$ JS-admissible.

(ii) $\tilde{f}$ JS-admissible, then $\tilde{f}'$ JS-admissible. Etc.

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Proposition

Consider

$$\tilde{f}(z) + \tilde{f}'(z) = \tilde{f}(qz) + \tilde{g}(z).$$

We have,

$$\tilde{g}(z) \textit{ JS-admissible} \quad \Longleftrightarrow \quad \tilde{f}(z) \textit{ JS-admissible.}$$

Poissonized Mean and Second Moment

Define

$$\tilde{f}_1(z) = e^{-z} \sum_{n\geq 0} \mathbb{E}(X_n) \frac{z^n}{n!}, \qquad \tilde{f}_2(z) e^{-z} \sum_{n\geq 0} \mathbb{E}(X_n^2) \frac{z^n}{n!}$$

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Then,

$$\tilde{f}_1(z) + \tilde{f}_1'(z) = \tilde{f}_1(qz) + 1$$
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Previous results show that $\tilde{f}_1(z), \tilde{f}_2(z)$ are JS admissible.

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More general:

$$\tilde{V}(z) = \tilde{f}_2(z) - \sum_{n\geq 0} \tilde{f}_1^{(n)}(z)^2 \frac{z^n}{n!}.$$

Then one obtains even identities!

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We start from,

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Applying Laplace transform,

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Define,

$$Q(s) := \sum_{l\geq 1}\left(1 - q^l s\right)$$

and $Q_\infty := Q(1)$.

Laplace and Mellin Transform (ii)

Set

$$\bar{f}(s) := \frac{\mathscr{L}[\tilde{f}(z); s]}{Q(-s)}, \qquad \bar{g}(s) := \frac{\mathscr{L}[\tilde{g}(z); s]}{Q(-s/q)}.$$

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Applying Mellin transform,

$$\mathscr{M}[\bar{f}(s); \omega] = \frac{\mathscr{M}[\bar{g}(s); \omega]}{1 - q^{\omega - 1}}.$$

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Applying Mellin transform,

$$\mathscr{M}[\bar{f}(s); \omega] = \frac{\mathscr{M}[\bar{g}(s); \omega]}{1 - q^{\omega-1}}.$$

From this, an asymptotic expansion of $\tilde{f}(z)$ as $z \to \infty$ is obtained via inverse Mellin transform and inverse Laplace transform.

Inverse Laplace Transform

Theorem (F., Hwang, Zacharovas)

Let the Laplace transform of $\tilde{f}(z)$ exist and be analytic in $\mathbb{C} \setminus (-\infty, 0]$.

Assume that

$$\mathscr{L}[\tilde{f}; s] = \mathcal{O}\left(|s|^{-\alpha}\right)$$

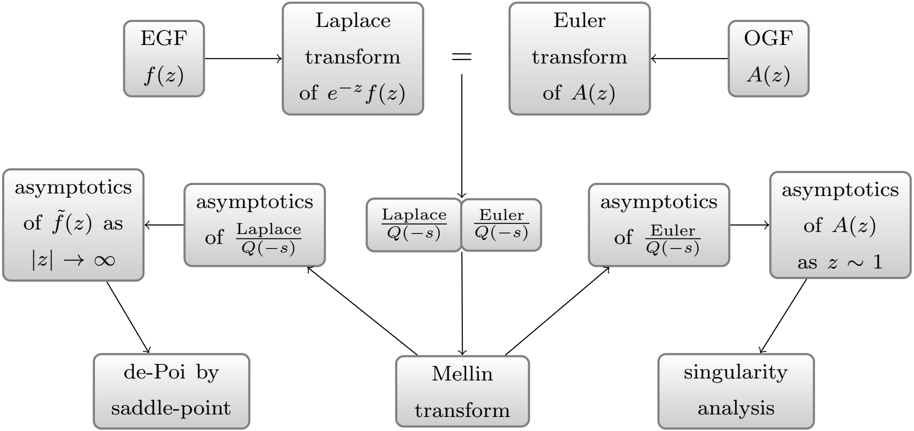
uniformly for $|s| \to 0$ and $|\arg(s)| \leq \pi - \epsilon$.

Then,

$$\tilde{f}(z) = \mathcal{O}\left(|z|^{\alpha-1}\right)$$

uniformly for $|z| \to \infty$ and $|\arg(z)| \leq \pi/2 - \epsilon$.

Our Approach vs. Flajolet-Richmond



Main Result

Theorem (F., Lee, Prodinger)

We have,

$$\mathrm{Var}(C_n) \sim \sum_k g_k n^{\chi_k},$$

where

$$g_k = \frac{Q_\infty}{L\Gamma(1+\chi_k)} \sum_{h,l,j\ge 0} \frac{(-1)^j q^{h+l+\binom{j+1}{2}}}{Q_h Q_l Q_j} \varphi(\chi_k; q^{h+j} + q^{l+j}).$$

Here, $\chi_k = 2k\pi i/L, L = \log(1/q), Q_j = \prod_{l=1}^{j}(1-q^l)$ *and*

$$\varphi(\chi; x) = \begin{cases} \pi(x^\chi - 1)/(\sin(\pi\chi)(x-1)), & \text{if } x \neq 1; \\ \pi\chi/\sin(\pi\chi), & \text{if } x = 1. \end{cases}$$

An Identity

Corollary (F., Lee, Prodinger)

We have,

$$\frac{Q_\infty}{L}\sum_{h,l,j\geq 0}\frac{(-1)^j q^{h+l+\binom{j+1}{2}}}{Q_hQ_lQ_j}\psi(q^{h+j}+q^{l+j})$$
$$=\frac{\pi^2}{6L^2}-\alpha-\beta+\frac{1}{12}-\frac{1}{L}\sum_{l\geq 1}\frac{1}{l\sinh(2l\pi^2/L)},$$

where

$$\psi(x)=\begin{cases}\log x/(x-1), & \text{if } x\neq 1;\\ 1, & \text{if } x=1.\end{cases}$$

We have a direct proof for this using tools from q-analysis.

Total Path Length (i)

T_n: total path length in symmetric digital search tree.

Theorem (F., Hwang, Zacharovas)

We have

$$\mathrm{Var}(T_n) \sim n(C_{\mathit{var}} + G(\log_2 n)),$$

where $G(z)$ is a 1-periodic function with zero average value and

$$C_{\mathit{var}} = \frac{Q_\infty}{L} \sum_{j,h,l \geq 0} \frac{(-1)^j 2^{-\binom{j+1}{2}}}{Q_j Q_h Q_l 2^{h+l}} \delta(2^{-j-h} + 2^{-j-l}),$$

where

$$\delta(x) := \begin{cases} (x - \log x - 1)/(x-1)^2, & \text{if } x \neq 1; \\ 1/2, & \text{if } x = 1. \end{cases}$$

Total Path Length (ii)

Variance of total path length was also derived by Kirschenhofer, Prodinger and Szpankowski with different expression for C_{var}.

Total Path Length (ii)

Variance of total path length was also derived by Kirschenhofer, Prodinger and Szpankowski with different expression for C_{var}.

To describe their expression we need:

- Let $[FG]_0$ denote the 0-th Fourier coefficient of the product of the two Fourier series $F(z)$ and $G(z)$.
- Put

$$F(z) = \frac{1}{L}\sum_{l\neq 0}\Gamma(-1-\chi_l)e^{2l\pi iz}$$

  and

$$H(z) = -\frac{1}{L}\sum_{l\neq 0}\left(1-\frac{\chi_l}{2}\right)\Gamma(-\chi_l)e^{2l\pi iz}.$$

$$
\begin{aligned}
C_{\mathsf{var}} = &-\frac{28}{3L}-\frac{39}{4}-2\sum_{l\ge1}\frac{l2^l}{(2^l-1)^2}+\frac{2}{L}\sum_{l\ge1}\frac{1}{2^l-1}+\frac{\pi^2}{2L^2}+\frac{2}{L^2}\\
&-\frac{2}{L}\sum_{l\ge3}\frac{(-1)^{l+1}(l-5)}{(l+1)l(l-1)(2^l-1)}\\
&+\frac{2}{L}\sum_{l\ge1}(-1)^l2^{-\binom{l+1}{2}}\left(\frac{L(1-2^{-l+1})/2-1}{1-2^{-l}}-\sum_{r\ge2}\frac{(-1)^{r+1}}{r(r-1)(2^{r+l}-1)}\right)\\
&-\frac{2Q(1)}{L}+\sum_{l\ge2}\frac{1}{2^lQ_l}\sum_{r\ge0}\frac{(-1)^r2^{-\binom{r+1}{2}}}{Q_r}Q_{r+l-2}\cdot\\
&\cdot\left(-\sum_{j\ge1}\frac{1}{2^{j+r+l+2}-1}\left(2^{l+1}-2l-4+2\sum_{i=2}^{l-1}\binom{l+1}{i}\frac{1}{2^{r+i-1}-1}\right)\right.\\
&\qquad+\frac{2}{(1-2^{-l-r})^2}+\frac{2l+2}{(1-2^{1-l-r})^2}-\frac{2}{L}\frac{1}{1-2^{1-l-r}}+\frac{2}{L}\sum_{j=1}^{l+1}\binom{l+1}{j}\frac{1}{2^{r+j}-1}\\
&\qquad\left.-2\sum_{j=2}^{l+1}\binom{l+1}{j}\frac{1}{2^{r+j-1}-1}+\frac{2}{L}\sum_{j=0}^{l+1}\binom{l+1}{j}\sum_{i\ge1}\frac{(-1)^i}{(i+1)(2^{r+j+i}-1)}\right)\\
&+\sum_{l\ge3}\sum_{r=2}^{l-1}\binom{l+1}{r}\frac{Q_{r-2}Q_{l-r-1}}{2^lQ_l}\sum_{j\ge l+1}\frac{1}{2^j-1}-2[FH]_0-[F^2]_0.
\end{aligned}
$$

Approximate Counting with m Counters (i)

Cichoń and Macyna:

Consider m counters. When counting the n-th object choose one uniform at random.

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Then,

$$D_n \stackrel{d}{=} C_{I_1}^{(1)} + \cdots + C_{I_m}^{(m)},$$

where

$$P(I_1 = n_1, \ldots, I_m = n_m) = \frac{1}{m^n}\binom{n}{n_1, \ldots, n_m}.$$

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Let $\tilde{f}_D, \tilde{f}_C$ denote Poisson mean of D_n and C_n. Similar, let $\tilde{V}_D$ and $\tilde{V}_C$ denote Poisson variance of D_n and C_n.

Approximate Counting with m Counters (ii)

Poisson model:

$$\tilde{f}_D(z) = m\tilde{f}_C(z/m),$$
$$\tilde{V}_D(z) = m\tilde{V}_C(z/m).$$

Approximate Counting with m Counters (ii)

Poisson model:

$$\tilde{f}_D(z) = m\tilde{f}_C(z/m),$$
$$\tilde{V}_D(z) = m\tilde{V}_C(z/m).$$

From this, we obtain the following result.

Theorem (F., Lee, Prodinger)

We have,

$$\mathbb{E}(D_n) \sim m\log_{1/q}(n/m) + mC_{mean} + mF(\log_{1/q}(n/m)),$$
$$\mathrm{Var}(D_n) \sim mC_{var} + mG(\log_{1/q}(n/m)),$$

where C_{mean}, C_{var} and $F(z), G(z)$ are as before.

Approximate Counting with m Counters (iii)

Another variant of approximate counting with m counters:

Consider m counters. Use a counter until it is increased; then cyclically move on to the next.

Approximate Counting with m Counters (iii)

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Approximate Counting with m Counters (iii)

Another variant of approximate counting with m counters:

Consider m counters. Use a counter until it is increased; then cyclically move on to the next.

D_n: sum of all counters after counting n objects.

Then,

$$D_n \stackrel{d}{=} X_n,$$

where

$$X_{n+m} \stackrel{d}{=} X_{I_n} + m.$$

X_n is the length of the leftmost path in random bucket digital search tree with bucket size m.

Approximate Counting with m Counters (iv)

Recurrence of moments:

$$f_{n+m} = \sum_{j=0}^{n} \binom{n}{j} q^j p^{n-j} f_j + g_n.$$

Approximate Counting with m Counters (iv)

Recurrence of moments:

$$f_{n+m} = \sum_{j=0}^{n} \binom{n}{j} q^j p^{n-j} f_j + g_n.$$

Poissonized variance $\tilde{V}(z)$ satisfies the differential function equation:

$$\sum_{i=0}^{m} \binom{m}{i} \tilde{V}^{(i)}(z) = \tilde{V}(qz) + \tilde{g}(z),$$

where

$$\tilde{g}(z) = \left(\sum_{i=0}^{m} \binom{m}{i} \tilde{f}_1^{(i)}(z) \right)^2 - \sum_{i=0}^{m} \binom{m}{i} \left(\tilde{f}_1(z)^2 \right)^{(i)}.$$

Approximate Counting with m Counters (v)

Theorem (F., Lee, Prodinger)

We have,

$$\mathrm{Var}(D_n) \sim \sum_k g_k n^{\chi_k},$$

where

$$g_k = \frac{1}{L\Gamma(1+\chi_k)} \int_0^\infty \frac{s^{\chi_k}}{Q(-s/q)^m} \left(p(s) + \int_0^\infty e^{-sz} \tilde{g}(z) \mathrm{d}z \right) \mathrm{d}s$$

and

$$p(s) = \frac{(s+1)^m - 1 - ms}{s^2}.$$