DISSERTATION

Limit Theorems in Metric Diophantine Approximation

ausgeführt zum Zwecke der Erlangung des akademischen Grades
eines Doktors der technischen Wissenschaften unter der Leitung von

Ao.Univ.Prof. Dipl.-Ing. Dr. Michael Drmota
Institut für Geometrie, E113

eingereicht an der Technischen Universität Wien
Fakultät für Technische Naturwissenschaften und Informatik

von

Michael Fuchs
Matr.Nr. 9525887
Reinprechtsdorferstraße 20/19, 1050 Wien

Wien, am 1. Februar 2002 ....

Kurzfassung (German)

Die Theorie der diophantischen Approximation, die sich aus dem Problem der Approximierbarkeit einer reellen Zahl durch rationale Zahlen entwickelt hat, ist wohl so alt wie die Mathematik selbst und erste Problemstellungen sowie Lösungsansätze können bis in die Antike zurückverfolgt werden. Der Beginn einer systematischen Entwicklung erfolgte aber erst im 19. Jahrhundert und am Anfang ging diese Hand in Hand mit der Entwicklung eines weiteren fumdamentalen Zweigs der Zahlentheorie nämlich der der Kettenbrüche. Während zunächst die Frage nach der Approximationsgüte aller reellen Zahlen bzw. aller irrationalen Zahlen im Vordergrund stand, wurde von Khintchine am Anfang des 20. Jahrhunderts ein neuer Weg eingeschlagen. Khintchine interessierte sich für das Approximationsverhalten einer typischen (oder besser zufällig gewählten) reellen Zahl. Diese neue Fragestellung war der Ausgangspunkt für die Entwicklung der *metrischen Theorie der diophantischen Approximation.*

Khintchine betrachtete die folgende diophantische Ungleichung

$$\left|x - \frac{p}{q}\right| \leq \frac{f(\log q)}{q^2}$$

in den Unbekannten p und q ($q > 0$) wobei f eine Funktion aus einer gegebenen Funktionenklasse bezeichnet und x zufällig aus dem Einheitsintervall gewählt wird. Unter Zuhilfenahme der metrischen Kettenbruchtheorie konnte Khintchine zeigen, dass die Menge aller jener x für welche die obige Ungleichung unendlich viele Lösungen besitzt einem 0-1 Gesetz unterliegt (dabei ist das zugrundeliegende Maß das Lebesgues Maß; siehe Satz 1.3.1 in Section 1.3.2). Von einem modernen wahrscheinlichkeitstheoretischen Standpunkt aus gesehen ist dieses Resultat nichts anderes als eine Anwendung des Lemmas von Borel-Cantelli für schwach abhängige Folgen von Zufallsvariablen. Khintchines Ergebnis wurde in den folgenden Jahren in viele Richtungen erweitert und verallgemeinert. Zum Beispiel konnte Erdös, ausgehend von Untersuchungen von LeVeque, für die Lösungsmenge der obigen diophantischen Ungleichung ein starkes Gesetz der grossen Zahlen beweisen (siehe Unterabschnitt 1.3.2), dass damit eine Verallgemeinerung des Satzes von Khintchine darstellte. Ziel dieser Arbeit war es geeignete

Funktionenklassen zu finden, für die es möglich ist stärkere wahrscheinlichkeitstheoretische Resultate als das Lemma von Borel-Cantelli oder das starke Gesetz der grossen Zahlen für die Lösungsmenge der obigen diophantischen Ungleichung abzuleiten (etwa einen zentralen Grenzwertsatz oder einen Satz vom iterierten Logarithmus).

Im ersten Kapitel dieser Arbeit werden bekannte Definitionen und Ergebnisse aus verschiedenen Teilgebieten der Mathematik gesammelt, die dann in den weiteren Kapiteln zur Ableitung der neuen Ergebnisse benötigt werden. Da diese Hilfsergebnisse im wesentlichen vier verschiedenen Zweigen der Mathematik angehören, ist das erste Kapitel in vier entsprechende Unterabschnitte eingeteilt. Im ersten Unterabschnitt werden Definitionen und Ergebnisse der Wahrscheinlichkeitstheorie zusammengestellt, wobei das Hauptinteresse auf die Theorie schwach abhängiger Folgen von Zufallsvariablen gelegt wird, da diese Theorie eng mit der statistischen Untersuchung der Lösungmenge der obigen diophantischen Ungleichung verknüpft ist (dies wird in Kapitel 2 und Kapitel 3 deutlich werden). Im zweiten und dritten Unterabschnitt werden Ergebnisse der metrischen Kettenbruchtheorie und metrischen diophantischen Approximation gesammelt, wobei starke, wahrscheinlichkeitstheoretische Resultate wie zentrale Grenzwertsätze und Sätze vom iterierten Logarithmus im Vordergrund stehen. Dabei werden im wesentlichen alle starken wahrscheinlichkeitsteoretischen Ergebnisse der metrischen diophantischen Approximation, die bis jetzt bekannt waren, formuliert und eine Vermutung von LeVeque in diesem Zusammenhang erwähnt, die auf das Jahr 1959 zurückgeht (siehe Seite 21). Im letzten Unterabschnitt wird der Körper der formalen Laurentreihen eingeführt, der viele Analogien zum Körper der reellen Zahlen besitzt (zum Beispiel eine Kettenbruchtheorie mit vielen analogen Resultaten zum reellen Fall) und alle Ergebnisse bezüglich metrischer diophantischer Approximation und metrischer Kettenbruchtheorie zusammengestellt, die bis jetzt in diesem Körper bewiesen wurden.

Im zweiten Kapitel werden im wesentlichen zwei Hauptziele verfolgt und dementsprechend ist dieses Kapitel in zwei Unterabschnitte eingeteilt. Im ersten Unterabschnitt (Section 2.1) wird die Vermutung von LeVeque, die im ersten Kapitel formuliert wurde, verallgemeinert und vollständig gelöst. Im zweiten Unterabschnitt (Section 2.2) werden alle vorher bekannten starken wahrscheinlichkeitstheoretischen Resulate der metrischen diophantischen Approximation erweitert, indem fast sichere Invarianzprinzipien und Invarianzprinzipien in Verteilung abgeleitet werden. Diese Resultate ziehen eine Reihe von Konsequenzen nach sich. Zum Beispiel erhält man, dass die Lösungsmenge der obigen diophantischen Ungleichung einen funktionalen zentralen Grenzwertsatz und Strassens Gesetz vom iterierten Logarithmus erfüllt. Beim Beweis der Resultate im zweiten Kapitel wird mit Hilfe der metrischen Kettenbruchtheorie das zahltentheoretische Problem in ein rein wahrscheinlichkeitstheoretisches übersetzt und dann entsprechende Resultate der Theorie der schwach abhängigen Folgen von Zufallsvariablen angewandt.

Im dritten Kapitel wird die Theorie der metrischen diophantischen Approximation im Körper der formalen Laurentreihen entwickelt und auch einige Resultate aus der metrischen Kettenbruchtheorie bewiesen. Insbesondere wird ersichtlich, dass alle Resultate der ersten beiden Kapiteln auch in entsprechender Form im Körper der formalen Laurentreihen gültig bleiben. Dabei ist metrische diophantische Approximation im Körper der formalen Laurentreihen viel einfacher als im reellen Fall, was auf die ultrametrische Struktur der Topologie zurückzuführen ist. Insbesondere hat diese zur Konsequenz, dass die Teilnenner in der Kettenbruchentwicklung (aufgefasst als Folge von Zufallsvariablen) eine unabhängige und identisch verteilte Folge darstellen (vergleiche mit Section 1.4.2). Dies bewirkt, dass an die Stelle der Theorie der schwach abhängigen Zufallsvariablen, die im reellen Fall benötigt wurde, die Theorie der unabhängigen Zufallsvariablen tritt. Betrachtet man allerdings für die Lösungsmenge der obigen diophantischen Ungleichung die Einschränkung, dass der Nenner einer Lösung in eine gegebene arithmetische Progression fallen muss, dann sind die involvierten Zufallsvariablen wieder schwach abhängig und nicht länger unabhängig, im Gegensatz zum reellen Fall, wo in beiden Situationen schwache Abhängigkeit vorlag. Dementsprechend stellen die beide Fälle - alle Lösungen auf der einen Seite und Lösungen mit Nennern in einer arithmetischen Progression auf der anderen Seite - im Körper der formalen Laurentreihen verschiedene Problemstellungen dar und werden im dritten Kapitel auch getrennt behandelt. Dabei befasst sich der erste Unterabschnitt (Section 3.1) mit dem Fall aller Lösungen, indem sich die Methoden gänzlich vom reellen Fall unterscheiden. Im zweiten Unterabschnitt (Section 3.2) wird auf die Situation von Lösungen mit Nennern in einer arithmetschen Progression eingegangen. In diesem Fall lassen sich im wesentlichen die Methoden aus dem reellen Fall übertragen und die Hauptaufgabe in diesem Abschnitt liegt mehr auf der Bereitstellung der Hilfssätze auf denen die Beweise im reellen Fall ruhen. Die eigentliche Methode verläuft dann sehr ähnlich zum reellen Fall und wird nur mehr skizziert und auf allfällige Vereinfachungen hingewiesen.

Wien, am 1. Februar 2002 Michael Fuchs

Abstract

The development of the theory of diophantine approximation started two centuries ago with the question to which extent a real number can be approximated by rationals. While in the beginning, results for all real numbers (resp. all irrational numbers) were derived, it was Khintchine in the beginning of the last century who investigated the question from a different angle. He was interested in the approximation behaviour of a typical (or better to say randomly chosen) real number. This was the starting point of the *metric theory of diophantine approximation*.

Khintchine considered the following diophantine inequality

$$\left| x - \frac{p}{q} \right| \leq \frac{f(\log q)}{q^2}$$

in integers p, q with $q > 0$, where f is a suitable function and x is randomly chosen in the unit interval. By using metric theory of continued fraction expansion, Khintchine proved a zero-one law for the set of elements for which the inequality possesses infinitely many solutions (with respect to Lebesgue measure; see Theorem 1.3.1 in Section 1.3.1 for details). From a modern probabilistic point of view, this result is nothing else but an application of the Borel-Cantelli lemma for dependent random variables. Khintchine's theorem was improved and extended in many directions. In this work, we are especially interested in assumptions on f under which strong probabilistic results like the central limit theorem or the law of the iterated logarithm can be obtained.

In the first chapter, we are going to collect known results that will form the framework for the new results obtained in the further chapters. Since these results are coming from four different areas, we decided to divide the chapter into four sections. In the first section, we start with notation and results from probability theory. Thereby, we especially put emphasis on probabilistic results for sequences of random variables that satisfy a weak dependence assumption since the investigation of the solution set of the above inequality is closely connected to this theory (as we will see in Chapter 2 and Chapter 3). Then in the second and third section, we recall some results from the metric theory of continued fraction expansion and the

metric theory of diophantine approximation, whereas, we focus on strong probabilistic results, like central limit theorems or laws of the iterated logarithm. Especially, we are going to mention a conjecture of LeVeque raised in 1959 (see page 21) concerning such strong probabilistic results. Finally in the last section of Chapter 1, we pass from the field of real numbers to the field of formal Laurent series over a finite base field (for a definition see Section 1.4.1) and briefly recall the metric theory of continued fraction expansion and metric theory of diophantine approximation developed in this field so far.

In the second chapter, we mainly concentrate on two goals; therefore, the chapter is divided into two sections. In the first section, we extend and answer LeVeque's conjecture stated in the first chapter. In the second section, we improve all theorems known before concerning strong probabilistic results (especially, all theorems stated in Section 1.3.2) by providing almost sure and distribution type invariance principles. These results entail several consequences e.g. a functional version of the central limit theorem and Strassen's version of the law of iterated logarithm.

In the third and final chapter, we develop the metric theory of diophantine approximation in the field of formal Laurent series. Especially, we will see that all results from the first and second chapter can be obtained in one form or the other in the formal Laurent series case as well. Thereby, metric theory in the field of formal Laurent series is much easier than in the real case since the involved valuation is an ultrametric one. This is the deeper reason for the sequence of partial quotients in the continued fraction expansion to be independent and identically distributed; in opposite, to the real number field where this is not the case (for details see Section 1.4.2). Therefore, passing from the real case to the formal Laurent series case means also passing from the theory of weakly dependent random variables to the theory of independent random variables. This turns out to be wrong if restrictions on the solution set of the above inequality are posed. Therefore, we divide the third chapter into two sections according to whether restrictions are considered or not. In the part without restrictions (Section 3.1) the methods in the Laurent series case considerably differ from the real case, while in the part with restrictions (Section 3.2) the methods are very similar and therefore, we will concentrate in this part more on providing the fundamental lemmas on which the proofs of the results in the real case rest and then, we will just outline the differences that occur when the methods from the real case are transferred to the Laurent series case.

Vienna, February 1, 2002 Michael Fuchs

Contents

Chapter 1

Introduction

In this chapter, we introduce notation that we are going to use throughout the work and collect known results that we will need in the forthcoming chapters. Thereby, we start with results from probability theory; then, we summarize some results from the metric theory of continued fraction expansion and the metric theory of diophantine approximation. Finally, we introduce the field of formal Laurent series over a finite base field and briefly recall its theory of continued fractions. The only new result in this chapter is an extension of an invariance principle for weakly dependent random variables which we are going to state and prove in the next section.

1.1 Notation and Results from Probability Theory

Throughout the section we shall denote by $(\Omega, \mathcal{A}, P)$ a fixed probability space. Furthermore, we use the standard notation $\Phi(x)$ for the distribution function of the normal law $\mathcal{N}(0, 1)$.

1.1.1 Limit Theorems for Independent Random Variables

It is well-known that an independent and identically distributed sequence of random variables satifies the central limit theorem if the variance is finite and positive. In the more general case of an independent but not necessary identically distributed sequence of random variables the same is true if in addition Lindeberg's condition holds.

Later, we will have to deal with independent sequences of random variables having infinite second moments. Also in this case, it is possible to get a stable limit behaviour if a suitable normalization is used. The following theorem gives a necessary and sufficient condition that an independent sequence of random variables is asymptotically normal with a suitable normalization and makes no use of any moment conditions at all. Before stating it, we need the following notation

Definition 1.1.1. Let X be a random variable and $c \geq 0$ a real constant. We denote by

$$X^{(c)}(\omega) = \begin{cases} X(\omega) & \text{if } |X(\omega)| < c \\ 0 & \text{otherwise} \end{cases}$$

the random variable X *truncated at* c.

Theorem 1.1.1 (Chapter IV, Theorem 21 in [22]). *Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables. Then, in order that there exist sequences of real numbers $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ with $a_n > 0$ and such that*

$$\sup_{x\in\mathbb{R}} \left| P\left[\frac{1}{a_n}\left(\sum_{k=1}^{n} X_k - b_n\right) < x\right] - \Phi(x)\right| \longrightarrow 0$$

it is necessary and sufficient that there exists a sequence of real numbers $(c_n)_{n\geq 1}$ with $c_n \longrightarrow \infty$ and

$$\sum_{k=1}^{n} P[|X_k| \geq c_n] \longrightarrow 0 \tag{1.1}$$

and

$$\frac{1}{c_n^2}\sum_{k=1}^{n} \mathbf{V}X_k^{(c_n)} \longrightarrow \infty. \tag{1.2}$$

Especially, if there exists a sequences $(c_n)_{n\geq 1}$ with the above properties then, we have

$$\begin{aligned} a_n^2 &= \sum_{k=1}^{n} \mathbf{V}X_k^{(c_n)} \\ b_n &= \sum_{k=1}^{n} \mathbf{E}X_k^{(c_n)}. \end{aligned}$$

1.1.2 Limit Theorems for Dependent Random Variables

In the last decades, many theorems for sequences of independent random variables were generalized to the case of weak dependence. Thereby, lots of different notations of weak dependence occur in literature and now, we are going to summarize few that we are going to use.

For a sequence $(X_n)_{n\geq 1}$ of random variables, we shall use $\mathcal{F}_a^b$ to indicate the σ-algebra generated by $(X_k)_{a\leq k\leq b}$.

Definition 1.1.2. Let $(X_n)_{n\geq 1}$ be a sequence of random variables. We are going to call the sequence

- $*$-*mixing* if there exists a function ϕ defined on the positive integers with $\phi \downarrow 0$ and

  $$|P(A \cap B) - P(A)P(B)| \leq \phi(n)P(A)P(B)$$

  for all $A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+n}^{\infty}, k \geq 1$, and $n \geq 1$.

- ϕ-*mixing* if there exists a function ϕ defined on the positive integers with $\phi \downarrow 0$ and

  $$|P(A \cap B) - P(A)P(B)| \leq \phi(n)P(A)$$

  for all $A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+n}^{\infty}, k \geq 1$, and $n \geq 1$.

- *strong mixing* if there exists a function ϕ defined on the positive integers with $\phi \downarrow 0$ and

  $$|P(A \cap B) - P(A)P(B)| \leq \phi(n)$$

  for all $A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+n}^{\infty}, k \geq 1$, and $n \geq 1$.

- *retarded strong mixing* if there exists a function ϕ defined on the positive integers with $\phi \downarrow 0$ and a non-negative real constant κ such that

  $$|P(A \cap B) - P(A)P(B)| \leq \phi(nk^{-\kappa}) \tag{1.3}$$

  for all $A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+n}^{\infty}, k \geq 1$, and $n \geq 1$.

Remark 1.1.1. It is clear that we have the following chain of inclusions for the above notations of weak dependence

$$\star\text{-mixing} \Rightarrow \phi\text{-mixing} \Rightarrow \text{strong mixing} \Rightarrow \text{retarded strong mixing.}$$

Next, we point out some elementary but useful lemmas.

Lemma 1.1.1 (Lemma 1.2.1 in [23]). *Let $(X_n)_{n\geq 1}$ be a $\star$-mixing sequence of random variables. Furthermore, let ξ and η be integrable random variables and ξ resp. η measurable over $\mathcal{F}_1^k$ resp. $\mathcal{F}_{k+n}^{\infty}$ for fixed $k, n \geq 1$. Then, we have*

$$|\mathbf{E}\xi\eta - \mathbf{E}\xi\mathbf{E}\eta| \leq \phi(n)\mathbf{E}|\xi|\mathbf{E}|\eta|. \tag{1.4}$$

Lemma 1.1.2 (Lemma 1.2.2 in [23]). *Let $(X_n)_{n\geq 1}$ be a ϕ-mixing sequence of random variables. Furthermore, let ξ and η be random variables with ξ resp. η measurable over $\mathcal{F}_1^k$ resp. $\mathcal{F}_{k+n}^{\infty}$ for fixed $k, n \geq 1$. Assume that $\mathbf{E}|\xi|^r < \infty$ and $\mathbf{E}|\eta|^s < \infty$ for integers $r, s > 1$ with $1/r + 1/s = 1$. Then, we have*

$$|\mathbf{E}\xi\eta - \mathbf{E}\xi\mathbf{E}\eta| \leq 2\phi(n)^{1/r}\|\xi\|_r\|\eta\|_s.$$

Lemma 1.1.3 (Lemma 7.2.1 in [24]). *Let $(X_n)_{n\geq 1}$ be a strong mixing sequence of random variables. Furthermore, let ξ and η be random variables with ξ resp. η measurable over $\mathcal{F}_1^k$ resp. $\mathcal{F}_{k+n}^\infty$ for fixed $k, n \geq 1$. Assume that $\mathbf{E}|\xi|^r < \infty$ and $\mathbf{E}|\eta|^s < \infty$ for integers $r, s, t > 1$ with $1/r + 1/s + 1/t = 1$. Then, we have*

$$|\mathbf{E}\xi\eta - \mathbf{E}\xi\mathbf{E}\eta| \leq 10\phi(n)^{1/t}\|\xi\|_r\|\eta\|_s.$$

In [24], Philipp and Stout proved almost sure invariance principles for sequences of random variables satisfying one of the previous mixing conditions. These results unified and extended many of the results known before in this direction. Here, we are interested in an extension of Theorem 7.1 in [24].

Theorem 1.1.2. *Let $(\xi_n)_{n\geq 1}$ be a sequence of random variables and f a measurable function on the set of sequences into the real line. Define*

$$\eta_n = f((\xi_k)_{k\geq n})$$

for $n \geq 1$ and

$$\eta_{l,n} = \mathbf{E}(\eta_n|\mathcal{F}_n^{n+l})$$

for $n, l \geq 1$ (where $\mathcal{F}_a^b$ denotes the σ-algebra generated by $(\xi_k)_{a\leq k\leq b}$). We shall assume of the function f and the sequence $(\xi_n)_{n\geq 1}$ that

$$\mathbf{E}\eta_n = 0.$$

Suppose that there exist constants $0 < \delta \leq 2$ and $C > 0$ such that

$$\mathbf{E}|\eta_n|^{2+\delta} \leq C\log n \tag{1.5}$$

and

$$\|\eta_n - \eta_{l,n}\|_{2+\delta} \leq Cl^{-(2+7/\delta)}$$

for all $n, l = 1, 2, 3, \ldots$. Moreover, suppose that

$$\mathbf{E}\left(\sum_{n\leq N}\eta_n\right)^2 \sim N \tag{1.6}$$

as $N \longrightarrow \infty$. Finally, assume that $(\xi_n)_{n\geq 1}$ satisfies a retarded strong mixing condition of the form (1.3) with

$$\kappa = \delta/(11 + 4\delta)$$

and

$$\phi(n) \ll n^{-168/(1+2/\delta)}.$$

Define a process $\{S(t), t \geq 0\}$ on the product probability space $(\Omega, \mathcal{A}, P) \times ([0,1], \mathcal{B}, \lambda)$ (here $\mathcal{B}$ denotes the σ-algebra of Borel sets and λ Lebesgue measure both restricted to the unit interval) by setting for $\omega \in \Omega$ and $x \in [0,1]$

$$S(t;\omega,x) := S(t) := \sum_{n \leq t} \eta_n(\omega).$$

Then, we have, as $t \longrightarrow \infty$,

$$S(t) - W(t) = o((t \log\log t)^{1/2}) \ \ a.s. \tag{1.7}$$

and

$$(P \times \lambda)\left[\frac{1}{\sqrt{t}} \sup_{\tau \leq t} |S(\tau) - W(\tau)| \geq \epsilon\right] \longrightarrow 0 \tag{1.8}$$

for all $\epsilon > 0$, whereas $\{W(t), t \geq 0\}$ denotes Brownian motion on the above probability space.

Furthermore, if we replace (1.6) by the following stronger assumption

$$\mathbf{E}\left(\sum_{k=1}^{n} \eta_k\right)^2 = n + O(n^{1-\delta/30}) \tag{1.9}$$

then, we obtain the stronger result

$$S(t) - W(t) \ll t^{1/2-\lambda} \ \ a.s., \ \ as\ t \longrightarrow \infty,$$

for each $\lambda < \delta/588$.

As already mentioned, this result is an extension of the cited result due to Philipp and Stout.

First of all, the result is much more explicit because the formulation "...we can redefine the process on a richer probability space..." used in [23] was avoided by providing an explicit probability space that is rich enough. Thereby, as it is stated in the theorem, it is sufficient to adjoin a uniformly distributed random variable, independent of the entire sequence $(\eta_n)_{n\geq 1}$. Since this achievment can be easily proved by combining the method in [23] with the approach of Dudley and Philipp in [3], we are not going to stress this point further (compare also with the remark made in [25]).

Second, the $2+\delta$-moments of the sequence $(\eta_n)_{n\geq 1}$ were assumed to be uniformly bounded in [24]. An inspection of the proof easily shows that this can be relaxed to (1.5).

Third, in [24] the theorem is only stated with the stronger assumption (1.9). In order to prove the theorem with assumption (1.6), we can essentially follow the proof in [24]. Therefore, we will briefly recall the proof idea of Theorem 7.1 in [24] (compare also with Chapter 2 in [24]) and then, we are going to point out the modifications which have to be made if (1.6) is used instead of (1.9).

The proof of Theorem 7.1 in [24] is based on the following quite simple result.

Lemma 1.1.4 (Lemma 2.1 in [24]). *Let $(y_k)_{k\geq 1}$ be a sequence of random variables and $(\mathcal{L}_k)_{k\geq 0}$ a sequence of nondecreasing σ-algebras such that y_k is measurable over $\mathcal{L}_k$ for all k (thereby $\mathcal{L}_0$ is the trivial σ-algebra). Assume that*

$$\sum_{j=0}^{\infty} \mathbf{E}|\mathbf{E}(y_{k+j}|\mathcal{L}_j)| < \infty$$

for $k \geq 1$. Then, we have, for all $k \geq 1$

$$y_k = Y_k + u_k - u_{k+1}$$

where $(Y_k, \mathcal{L}_k)_{k\geq 1}$ is a martingale difference sequence and

$$u_k = \sum_{j=0}^{\infty} \mathbf{E}(y_{k+j}|\mathcal{L}_{k-1})$$

for all $k \geq 1$.

The proof of Theorem 7.1 in [24] proceeds now in two steps. In a *first step* Lemma 1.1.4 is used in order to approximate the sequence $(\eta_k)_{k\geq 1}$ by a martingale difference sequence.

But before this can be done, we have to split up the sequence η_k in small and big blocks, whereas a small block is always followed by a big one. Let us denote by y_k the sum of the η_j contained in the k-th big block and by z_k the sum of the η_j contained in the k-th small block respectively. Furthermore, we denote by M_N the number of the block in which η_N is contained (can be either a small or a big one). The next question is: how to choose the size of the small blocks? In fact, this is a crucial point since the size should not be too large on the one hand because then z_k is not negligible any more and the size should not be too small on the other hand because then the u_k obtained from Lemma 1.1.4 are not negligible. If the size is properly chosen then we get

$$\sum_{k\leq N} \eta_k \approx \sum_{k\leq M_N} y_k + z_k \approx \sum_{k\leq M_N} y_k$$

and we can apply Lemma 1.1.4 in order to approximate y_k by a martingale difference sequence

$$\sum_{k\leq M_N} y_k = \sum_{k\leq M_N} Y_k + u_k - u_{k-1} \approx \sum_{k\leq M_N} Y_k. \tag{1.10}$$

Consequently, we have approximated η_k by a martingale difference sequence

$$\sum_{k \leq N} \eta_k \approx \sum_{k \leq M_N} Y_k.$$

Now, in the *second step*, we apply the martingale version of the Skorokhod representation theorem (see Theorem 4.3 in [29]): there exist a seqence $(T_k)_{k \geq 1}$ of nonnegative random variables such that

$$\sum_{k \leq M_N} Y_k = W\left(\sum_{k \leq M_N} T_k\right) \quad \text{a.s.} \tag{1.11}$$

Furthermore, we use (1.9) in order to prove a strong law of large numbers with error term for the sequence y_k^2

$$\sum_{k \leq M_N} y_k^2 = N + \mathrm{O}(N^{1-2\alpha}) \quad \text{as} \quad n \longrightarrow \infty \tag{1.12}$$

where α is a suitable constant. This together with the approximation in (1.10) can be used to obtain

$$\sum_{k \leq M_N} Y_k^2 = N + \mathrm{O}(N^{1-2\alpha}) \quad \text{as} \quad n \longrightarrow \infty. \tag{1.13}$$

Now, the last result along with (1.11) in turn implies

$$\sum_{k \leq M_N} T_k = N + \mathrm{O}(N^{1-2\alpha}) \quad \text{as} \quad n \longrightarrow \infty. \tag{1.14}$$

Finally, by combining (1.11) and (1.14), we get

$$\begin{aligned} \sum_{k \leq N} Y_k &= W\left(\sum_{k \leq M_N} T_k\right) \\ &= W\left(N + \mathrm{O}(N^{1-2\alpha})\right) \\ &= W(N) + \mathrm{O}(N^{1/2-\lambda}) \quad \text{a.s.} \end{aligned}$$

for a suitable constant λ. Thus, if we define

$$S^*(t) = \sum_{k \leq M_{[t]}} Y_k$$

we have

$$S^*(t) - W(t) \ll t^{1/2-\lambda} \quad \text{a.s.} \tag{1.15}$$

as $t \longrightarrow \infty$ and because of the first step we have the same result for $S(t)$ as well.

So, if we replace (1.9) by (1.6) then the first step remains unchanged and we have to take care only about the second step. Of course, we again use the martingal version of the Skorokhod representation theorem. Furthermore, by using the same proof method as in [24], we easily obtain instead of (1.12), (1.13), and (1.14) the following lemmas

Lemma 1.1.5. *As* $N \longrightarrow \infty$

$$\sum_{k=1}^{M_N} y_k^2 \sim N \ \ a.s.$$

Lemma 1.1.6. *As* $N \longrightarrow \infty$

$$\sum_{k=1}^{M_N} Y_k^2 \sim N \ \ a.s.$$

Lemma 1.1.7. *As* $N \longrightarrow \infty$

$$\sum_{k=1}^{M_N} T_k \sim N \ \ a.s.$$

Then, we use some ideas due to Strassen [28] in order to get instead of (1.15) the following result

Lemma 1.1.8. *As* $t \longrightarrow \infty$

$$S^*(t) - X(t) = o((t \log\log t)^{1/2}) \ \ a.s.$$

Proof. This can easily be seen by Lemma 1.1.7 and the classical method of Strassen (see page 217 in [28]). □

Finally, the last lemma together with the first step gives (1.7). For the proof of (1.8), we point out that the proof of Lemma 7.3.5 in [24] implies (also if (1.9) is replaced by (1.6))

Lemma 1.1.9. *As* $N \longrightarrow \infty$

$$\sum_{k=1}^{M_N} y_k^2 - \mathbf{E} y_k^2 \ll N^{1-2\alpha} \ \ a.s. \tag{1.16}$$

and

$$\sum_{k=1}^{M_N} \mathbf{E} y_k^2 \sim N. \tag{1.17}$$

This result allows us to prove

Lemma 1.1.10. *As* $N \longrightarrow \infty$

$$\max_{k \le N} \left| \frac{\sum_{j \le M_k} y_j^2}{N} - \frac{k}{N} \right| \xrightarrow{P} 0. \tag{1.18}$$

Proof. Let $\epsilon > 0$ be a given real number. It is immediate from (1.17) that we have

$$\max_{k \le N} \left| \frac{\sum_{j \le M_k} \mathbf{E} y_j^2}{N} - \frac{k}{N} \right| < \epsilon/2$$

for N large enough. Furthermore, observe by (1.16) that for each $\delta > 0$ there exist a set E with measure at least $1 - \delta$ and a constant κ such that

$$\left| \sum_{k=1}^{M_N} y_k^2 - \mathbf{E} y_k^2 \right| \le \kappa N^{1-2\alpha}$$

for all $N \ge 1$ and all elements in E. Hence

$$\max_{k \le N} \left| \frac{\sum_{j \le M_k} y_j^2 - \mathbf{E} y_j^2}{N} \right| \le \frac{\kappa}{N^{2\alpha}} < \epsilon/2$$

for all elements in E and N large enough. Therefore, we get

$$P\left(\left[\max_{k \le N} \left| \frac{\sum_{j \le M_k} y_j^2}{N} - \frac{k}{N} \right| \ge \epsilon \right] \cap E \right) \longrightarrow 0$$

as $N \longrightarrow \infty$. By using the trivial inequality $P(A \cap B) \ge P(A) + P(B) - 1$ the result is easily obtained. ☐

Furthermore, we get a similar result for the sequences Y_k^2 and T_k

Lemma 1.1.11. *As* $N \longrightarrow \infty$

$$\max_{k \le N} \left| \frac{\sum_{j \le M_k} Y_j^2}{N} - \frac{k}{N} \right| \xrightarrow{P} 0 \tag{1.19}$$

and

$$\max_{k \le N} \left| \frac{\sum_{j \le M_k} T_j}{N} - \frac{k}{N} \right| \xrightarrow{P} 0. \tag{1.20}$$

Proof. Since, as $N \longrightarrow \infty$,

$$\sum_{k=1}^{M_N} Y_k^2 - y_k^2 \ll N^{1-2\alpha} \text{ a.s.}$$

(this follows by (7.4.3) and (7.4.4) in [24] which are still true if (1.9) is replaced by (1.6)), we obtain (1.19) by using ideas of the proof of the last lemma and (1.18). The same is true for (1.20) because of

$$\sum_{k=1}^{M_N} T_k = \sum_{k=1}^{M_N} (T_k - \mathbf{E}(T_k|\mathcal{P}_{k-1})) + \sum_{k=1}^{M_N} (\mathbf{E}(Y_k^2|\mathcal{L}_{k-1}) - Y_k^2) + \sum_{k=1}^{M_N} Y_k^2$$

and

$$\sum_{k=1}^{M_N} (T_k - \mathbf{E}(T_k|\mathcal{P}_{k-1})) \ll N^{1-2\alpha} \text{ a.s.}$$
$$\sum_{k=1}^{M_N} (\mathbf{E}(Y_k^2|\mathcal{L}_{k-1}) - Y_k^2) \ll N^{1-2\alpha} \text{ a.s.}$$

where the σ-algebras $\mathcal{P}_k$ and $\mathcal{L}_k$ are defined in [24] (this follows by Lemma 7.4.4, (7.5.1) and the proof of Lemma 7.5.1 in [24] which are still true if (1.9) is replaced by (1.6)). □

We use this to replace (1.15) by the following result

Lemma 1.1.12. *As* $t \longrightarrow \infty$

$$P\left[\frac{1}{\sqrt{t}} \sup_{\tau \le t} |S^*(\tau) - X(\tau)| \ge \epsilon\right] \longrightarrow 0$$

for all $\epsilon > 0$.

Proof. First we observe, as $t \longrightarrow \infty$,

$$P\left[\sup_{\tau \le t} \left| \frac{\sum_{j \le M_{[\tau]}} T_j}{t} - \frac{\tau}{t} \right| \ge \delta\right] \longrightarrow 0 \tag{1.21}$$

which is easily obtained from (1.20). Next consider

$$P\left[\frac{1}{\sqrt{t}} \sup_{\tau \le t} |S^*(\tau) - X(\tau)| \ge \epsilon\right] = P\left[\sup_{\tau \le t} \left| X\left(\frac{\sum_{j \le M_{[\tau]}} T_j}{t}\right) - X\left(\frac{\tau}{t}\right) \right| \ge \epsilon\right]$$
$$\le P\left[\sup_{\tau \le t} \left| \frac{\sum_{j \le M_{[\tau]}} T_j}{t} - \frac{\tau}{t} \right| \ge \delta\right] + P\left[\sup_{\tau_2 \le 1} \sup_{|\tau_1 - \tau_2| < \delta} |X(\tau_1) - X(\tau_2)| \ge \epsilon\right].$$

The right hand side of the above inequality converges to zero, because of (1.21) on the one hand and the a.s. continuity of the Brownian motion paths on the other hand. This proves the result. □

The last lemma together with the first step in the proof of Theorem 7.1 in [24] immediatly gives (1.8). Therefore, we are done with the proof of our Theorem 1.1.2.

1.1.3 The Gaal-Koksma Method

In this section, we provide a theorem due to Gaal-Koksma that is a standard tool in metric number theory and is often referred as the *Gaal-Koksma method.*

Theorem 1.1.3 (Theorem 1.155 in [4]). *Let $F(M,N), M,N = 0,1,\ldots$ be a double sequence of nonnegative random variables with finite p-th moments where $p \geq 1$. Suppose that $F(M,0) = 0$ and*

$$F(M,N) \leq F(M,N') + F(M+N', N-N')$$

for all $M,N = 0,1,\ldots$ and all $0 \leq N' \leq N$. Furthermore, suppose

$$\mathbf{E}F(M,N)^p \ll \Psi(N)$$

uniformly in M where $\Psi(N)$ is a function defined on the nonnegative integers such that $\Psi(N)/N$ is nondecreasing. Then, we have

$$F(0,N)^p \ll \Psi(N)(\log N)^{p+1+\epsilon} \quad a.s.$$

for all $\epsilon > 0$.

1.2 Some Results from Metric Continued Fractions Theory

Before starting, we fix some notation concerning the theory of continued fractions that we are going to use throughout the work. First, we use the standard notation $x = [a_0, a_1, \ldots]$ for the continued fraction expansion of $x \in \mathbb{R}$ and denote by

$$\frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k]$$

the k-th convergent. Furthermore, we put

$$\varphi_k = [a_{k+1}, a_{k+2}, a_{k+3}, \ldots] + [0, a_k, a_{k-1}, \ldots, a_0],$$

and

$$\xi_k = [0, a_{k+1}, a_{k+2}, \ldots].$$

In the following, we are interested in the metric theory of continued fraction expansion. Therefore, we denote by $([0,1], \mathcal{B}, \lambda)$ the probability space that is obtained by restricting Lebesgue measure to the unit interval. Especially, the above sequences can be viewed as random variables on this probability space. In the following, we always want to make use of this point of view.

1.2.1 The Gauss-Kuzmin Theorem and Applications

The development of metric theory of continued fractions started in 1928 when Kuzmin recovered and sharpened a theorem due to Gauss (see [15]). This theorem is now known as the *Gauss-Kuzmin Theorem.*

Theorem 1.2.1 (Gauss-Kuzmin). *For $t \in [0,1]$ define*

$$m_k(t) := \lambda\left(\{x \in [0,1] | \xi_k \leq t\}\right).$$

Then, we have

$$m_k(t) = \frac{\log(1+t)}{\log 2}(1 + O(q^k)), \quad \textit{as} \quad k \longrightarrow \infty,$$

where $q < 1$ and this hold uniformly in t.

This important result entails several consequences. First, we immediatly get the asymptotic distribution of the partial quotients in the continued fraction expansion

Corollary 1.2.1. *We have, as $k \longrightarrow \infty$,*

$$\lambda\left(\{x \in [0,1] | a_k = r\}\right) = \frac{1}{\log 2}\log\left(1 + \frac{1}{r(r+2)}\right) + O(q^k).$$

Furthermore, the method of proof of Theorem 1.2.1 can be used to prove asymptotic independence of certain events. Especially, this implies

Corollary 1.2.2. *The sequence of random variables $(a_k)_{k \geq 1}$ on the probability space $([0,1], \mathcal{B}, \lambda)$ is $*$-mixing.*

Therefore, we can investigate the asymptotic behaviour of the sequence of partial quotients by applying the theory of $\star$-mixing sequences of random variables. The first result in this direction was already obtained by Khintchine before the theory of $\star$-mixing sequences of random variables was developed. Khintchine proved the following strong law of large numbers (see [14])

Theorem 1.2.2 (Khintchine). *Let f be a function on the positive integers with $f(r) = O(r^p)$ where $0 \leq p < 1/2$. Then, we have for almost all $x \in [0,1]$*

$$\lim_{n \longrightarrow \infty} \frac{1}{n}\sum_{k=1}^{n} f(a_k) = C(f)$$

with

$$C(f) = \frac{1}{\log 2}\sum_{r=1}^{\infty} f(r)\log\left(1 + \frac{1}{r(r+2)}\right).$$

As the theory of $\star$-mixing sequences of random variables became more and more powerful, the above theorem of Khintchine was improved several times. The final, best possible result was obtained by Szüsz in [32] by providing the following law of the iterated logarithm

Theorem 1.2.3. *Let f be as in Theorem 1.2.2. Then, we have for almost all $x \in [0,1]$*

$$\limsup_{n \longrightarrow \infty} \frac{\sum_{k=1}^{n} f(a_k) - nC(f)}{\sqrt{n \log \log n}} = \sigma$$

where $\sigma > 0$ is a suitable constant.

Especially, if we choose $f(r) = \log r$ in the above theorem, we obtain asymptotic results for the following sequence

$$\sum_{k=1}^{n} \log a_k = \log \prod_{k=1}^{n} a_k. \tag{1.22}$$

This sequence is closely related to the sequence $(\log q_n)_{n \geq 1}$ which was also already investigated by Khintchine and at the same time by Levy (see [19]). They independently obtained the following strong law of large numbers now known as *Khintchine-Levy Theorem*

Theorem 1.2.4 (Khintchine-Levy). *For almost all $x \in [0,1]$ we have*

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \log q_n = \gamma \quad a.s.$$

where $\gamma = \pi^2/(12 \log 2)$ is the Khintchine-Levy constant.

Also this theorem was improved several times by providing more and more sharper error terms. Here, the best possible result was finally obtained by Gordin and Reznik (see [10]).

Theorem 1.2.5. *For almost all $x \in [0,1]$ we have*

$$\limsup_{n \longrightarrow \infty} \frac{|\log q_n - n \log \gamma|}{\sqrt{n \log \log n}} = \sigma,$$

where $\sigma > 0$ and γ is again the Khintchine-Levy constant.

1.2.2 An Extension of the Gauss-Kuzmin Theorem

In [30], Szüsz proved the following extension of the Gauss-Kuzmin Theorem

Theorem 1.2.6. *For $t \in [0,1]$ and $a, b, r \in \mathbb{N}_0, r \geq 1$ define*

$$m_k(a,b,t) := \lambda\left(\{x \in [0,1] | q_{k-1} \equiv a \ (r), q_k \equiv b \ (r), \xi_k \leq t\}\right).$$

Then, we have

$$m_k(a,b,t) = \begin{cases} \frac{1}{C(r)}\frac{\log(1+t)}{\log 2}(1+O(q^k)) & \text{if } (a,b,r)=1, \\ 0 & \text{if } (a,b,r)\neq 1, \end{cases}$$

where $C(r) = r^2\prod_{p|r}\left(1-\frac{1}{p^2}\right)$, $q<1$ *is a constant and the constant implied in the error term depends only on* r.

Furthermore, he applied his result to investigate the asymptotic behaviour of sequences of random variables that are defined with help of a_k and $q_k, q_{k-1} \bmod r$. To be more precise, he proved the following extension of Theorem 1.2.2

Theorem 1.2.7. *Let* $f(k_1,k_2,r)$ *be a function which is periodic* $\operatorname{mod} r, r\geq 1$ *in the first two coordinates. Furthermore, assume*

$$f(k_1,k_2,r) = O(r^p)$$

with $0\leq p<1/2$. *Then, we have for almost all* $x\in[0,1]$

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\sum_{k=1}^{n} f(q_{k-1},q_k,a_k) = \frac{1}{C(r)\log 2}\sum_{(a,b,r)=1, 0\leq a,b<r} \sum_{r=1}^{\infty} f(a,b,r)\log\left(1+\frac{1}{r(r+2)}\right)$$

where

$$C(r) = r^2\prod_{p|r}\left(1-\frac{1}{p^2}\right).$$

By using Gaal-Koksma method, it is straightforward to sharpen the above theorem by providing an error term. In doing this, we obtain for almost all $x\in[0,1]$

$$\sum_{k=1}^{n} f(q_{k-1},q_k,a_k) = \frac{n}{C(r)\log 2}\sum_{(a,b,r)=1, 0\leq a,b<r} \sum_{r=1}^{\infty} f(a,b,r)\log\left(1+\frac{1}{r(r+2)}\right) + \mathrm{O}(n^{1/2}(\log n)^{3/2+\epsilon})$$

where $\epsilon>0$ is an arbitrary but fixed constant. Whether a stronger result (for instance a law of the iterated logarithm) is true or not is up to our knowledge not known.

Finally, we would like to mention a result which is closely related to Theorem 1.2.6 and actually, Szüsz obtained the result in [31] by applying a similar proof method

Theorem 1.2.8. *For $t \geq 2$ and $a, b, r \in \mathbb{N}_0, r \geq 1$ define*

$$m_k(a,b,t) := \lambda\left(\{x \in [0,1] | q_{k-1} \equiv a\ (r), q_k \equiv b\ (r), \varphi_k \geq t\}\right).$$

Then, we have

$$m_k(a,b,t) = \begin{cases} \frac{1}{C(r)\log 2} t^{-1}(1 + O(q^k)) & \text{if } (a,b,r) = 1, \\ 0 & \text{if } (a,b,r) \neq 1, \end{cases}$$

where $C(r) = r^2 \prod_{p|r}\left(1 - \frac{1}{p^2}\right)$, $q < 1$ is a constant, and the constant implied in the error term again depends only on r.

1.3 Some Results from Metric Diophantine Approximation

The metric theory of diophantine approximation concerns about statistical properties of the solutions of the following diophantine inequality

$$\left| x - \frac{p}{q} \right| \leq \frac{f(\log q)}{q^2} \tag{1.23}$$

where f is a function satisfying proper assumptions and x is an element chosen randomly in the unit interval. In this work, we are especially interested in restrictions on f under which strong results like the central limit theorem or the law of the iterated logarithm can be obtained for the set of solutions.

1.3.1 A Theorem of Khintchine and Extensions

The first result in the metric theory of diophantine approximation was proved by Khintchine in [13]. In this paper, Khintchine used continued fraction theory in order to derive a zero-one law for the set of x with infinitely many solutions $\langle p, q\rangle, q > 0$ of (1.23) and furthermore, the result was explicit in the sense that he gave a sufficient and necessary property for f to satisfy that this set has probability one. The result reads as follows

Theorem 1.3.1 (Khintchine). *Let f be a positive, non-increasing, real-valued function defined on the non-negative real numbers. Then, the inequality (1.23) possesses a.s. infinitely many solutions or a.s. finitely many solutions according to whether the following series*

$$\sum_{k=1}^{\infty} \frac{f(\log k)}{k}$$

diverges or converges.

From a probabilistic point of view, this theorem is nothing else but the Borel-Cantelli lemma for weakly dependent random variables.

Khintchine's Theorem was improved and extended in many directions. For instance, one can ask whether a similar result to Khintchine's Theorem is true or not if only solutions of (1.23) are taken into account that have denominator resp. enumerator contained in special sets. Thereby, the first restriction which occured in literature was to consider only solutions that have denominators contained in an arithmetic progression. That in this case, the following Khintchine type theorem is true was observed by Szüsz in [31]

Theorem 1.3.2. *Let f be a positive, non-increasing, real-valued function defined on the non-negative real numbers and $s, r \in \mathbb{N}_0, r \geq 1$. Then, the inequality (1.23) possesses a.s. infinitely many solutions that have denominators q with $q \equiv s\ (r)$ or a.s. finitely many solutions that have denominators q with $q \equiv s\ (r)$ according to whether the following series*

$$\sum_{k=1}^{\infty} \frac{f(\log k)}{k}$$

diverges or converges.

Another possibility to extend Khintchine's Theorem is to take a closer look on the set of solutions of (1.23). Therefore, define

$$T_n(x) := \#\{\langle p, q\rangle | 1 \leq q \leq n, p/q \text{ is a solution of } (1.23)\}.$$

The statistical behaviour of the above sequence of random variables was investigated by several authors. Most of the results obtained were strong laws of large numbers with error terms. Only few authors were interested in proving stronger results like the central limit thoerem or the iterated logarithm law. Of course, such strong theorems are only possible to obtain if the class of functions is more restricted than in Khintchine's Theorem. In the next section, we briefly outline what is known concerning such strong results.

1.3.2 The Theorems of LeVeque and Philipp

Suppose f is a positive, non-increasing, real-valued function defined on the non-negative real numbers satisfying the following conditions

$$f(x) = \mathrm{O}(x^{-1}), f'(x) = \mathrm{O}(x^{-2}), \text{ as } x \longrightarrow \infty; \quad \sum_{k=1}^{\infty} f(k) = \infty. \qquad (1.24)$$

Furthermore, we put

$$F(n) = \sum_{k=1}^{n} \frac{f(\log k)}{k}.$$

Under the above restrictions on f, LeVeque claimed in [17] that he had proved the following result

We have

$$\lim_{n\to\infty}\left[T_n < \frac{12}{\pi^2}F(n) + \omega\left(\frac{12}{\pi^2}F(n)\right)^{1/2}\right] = \Phi(\omega).$$

LeVeque was little bit puzzled that the sequence of random variables $(T_n)_{n\geq 1}$ concentrates at $12/\pi^2 F(n)$, because from a heuristic point of view one would expect that the concentration takes place at $2F(n)$. This inconsistency was explained by Erdős who pointed out that LeVeque actually considered the following sequences of random variables

$$T_n^{(1)} := \#\{\langle p,q\rangle | 1 \leq q \leq n, (p,q) = 1, p/q \text{ is a solution of } (1.23)\}$$

instead of $(T_n)_{n\geq 1}$.

Therefore, LeVeque wrote a second paper [18] in order to correct his result and to investigate the statistical behaviour of the sequence he always had in mind, namely $(T_n)_{n\geq 1}$. His attempt in proving a central limit theorem for this sequence of random variables was not sucessful but he managed to extend the result obtained in the first paper by proving a theorem that he called an approximation to a central limit theorem for the sequence $(T_n)_{n\geq 1}$. In order to state it, fix an integer $d \geq 1$ and define a sequence of random variables as

$$T_n^{(d)} := \#\{\langle p,q\rangle | 1 \leq q \leq n, (p,q) \leq d, p/q \text{ is a solution of } (1.23)\}$$

Then the result is as follows

Theorem 1.3.3. *Let f be a function satisfying (1.24). Then, we have*

$$\lim_{n\to\infty}\Bigg[T_n^{(d)} < 2\Bigg(1 - \frac{6}{\pi^2}\sum_{i=d+1}^{\infty} i^{-2}\Bigg)F(n) + \omega\left(\left(\frac{12}{\pi^2}\sum_{i=1}^{d}\frac{2i-1}{i^2}\right)F(n)\right)^{1/2}\Bigg] = \Phi(\omega).$$

Furthermore, for the sequence $(T_n)_{n\geq 1}$ LeVeque obtained in [18] the following theorem

Theorem 1.3.4. *Let $f(x)$ be a positive, non-increasing, real-valued function defined on the non-negative real numbers which tends to 0 as x tends to infinity. Then, we have for almost all $x \in [0,1]$*

$$T_n(x) \sim F(n).$$

Without the assumption that $f(x)$ is tending to 0 as x tends to infinity, the last theorem would extend Khintchine's Theorem. Since this assumption seemed to be only technical, LeVeque conjectured that it can be dropped.

So, LeVeque's paper [18] left open two problems

- Problem 1: is the last theorem still valid, if the assumption that $f(x)$ tends to 0 as x tends to infinity is dropped?
- Problem 2: does the sequence of random variables $(T_n)_{n\geq 1}$ satisfy a central limit theorem for f satisfying (1.24)?

While Problem 1 was solved by Erdős in [5], the second problem remained unsolved.

Few years after LeVeque's paper, Philipp considerably improved the case $d = 1$ by relaxing the conditions on f, adding a law of the iterated logarithm, and using Theorem 1.2.6 in order to take only solutions of (1.23) into account that have denominators contained in an arithmetic progression (see [23]). In detail, now suppose f to be a positive, real-valued function defined on the non-negative real numbers satisfying the following conditions

$$f \downarrow 0, \quad \textstyle\sum_{k=1}^{\infty} f(k) = \infty, \tag{1.25}$$

$$\textstyle\sum_{k=1}^{n} f(k)k^{-\delta_1} \ll \left(\sum_{k=1}^{n} f(k)\right)^{1/2-\delta_2}, \tag{1.26}$$

where $0 < \delta_1, \delta_2 < 1/2$. Furthermore, we put

$$F(n) = \sum_{k=1}^{n} \frac{f(\log k)}{k}.$$

Consider the following sequence of random variables

$$X_n^{(1)}(x) = \#\{\langle p, q\rangle | 1 \leq q \leq n, q \equiv s\ (r), (p, q) = 1, \\ p/q \text{ is a solution of (1.23)}\},$$

where $r \geq 1, s$ are arbitrary integers. Then, the result of Philipp reads as follows

Theorem 1.3.5. *We have*

$$\lim_{n\to\infty} \lambda\left[X_n^{(1)} < \sigma F(n) + \omega(\sigma F(n))^{1/2}\right] = \Phi(\omega),$$

and

$$\limsup_{n\to\infty} \frac{|X_n^{(1)} - \sigma F(n)|}{\sqrt{2\sigma F(n) \log\log F(n)}} = 1,$$

where

$$\sigma = \frac{12}{\pi^2}\frac{r\varphi((s,r))}{C(r)(s,r)}, \quad C(r) = r^2 \prod_{p|r}\left(1 - \frac{1}{p^2}\right).$$

Furthermore in [23], Philipp outlined a proof of LeVeque's second problem. However, there is an uncorrectable mistake in this sketch and therefore, LeVeque's second problem also remained unsolved after Philipp's paper. In chapter 2, we are going to give a proof of LeVeque's second conjecture.

1.4 The Field of Formal Laurent Series over a Finite Base Field

In this section, we introduce the field of formal Laurent series over a finite base field. This field possesses lots of similarities to the real number field, especially, there is a theory of continued fractions. While diophantine approximation results were proved by several authors in the formal Laurent series case, too (see for instance [16] or [27]), the theory of metric diophantine approximation was considered only by few authors. We will use Chapter 3 in order to develop the metric theory of diophantine approximation in this field.

1.4.1 Definition and Basic Results

By $\mathbb{F}_q$, we denote in the following the finite field with $q = p^t, p \in \mathbb{P}, t \geq 1$ elements. Furthermore, we use the standard notation $\mathbb{F}_q((T^{-1}))$ for the field of formal Laurent series over $\mathbb{F}_q$, which is the set of all formal sums

$$\alpha = \sum_{k \leq n} a_k T^k, \quad a_k \in \mathbb{F}_q, a_n \neq 0.$$

Clearly, the polynomial ring and the field of rational functions are contained in $\mathbb{F}_q((T^{-1}))$, whereas we have the following chain of inclusions

$$\mathbb{F}_q[T] \subset \mathbb{F}_q(T) \subset \mathbb{F}_q((T^{-1})).$$

Throughout this work, we write $\alpha, \beta, \ldots$ for the elements of $\mathbb{F}_q((T^{-1}))$ and $P, Q, \ldots$ for the elements of $\mathbb{F}_q[T]$. We define a valuation on $\mathbb{F}_q((T^{-1}))$ by $|\alpha| = q^n$ for $\alpha \neq 0$ resp. $|0| = 0$. It is easy to see that $\mathbb{F}_q((T^{-1}))$ endowed with this evaluation is an ultrametric space.

There is a continued fractions theory in $\mathbb{F}_q((T^{-1}))$ with polynomials playing the role of integers (see [27]). As in the classical theory, we denote by $\alpha = [A_0, A_1, A_2, \ldots]$ the continued fraction expansion of $\alpha \in \mathbb{F}_q((T^{-1}))$. Furthermore, we use

$$\frac{P_k}{Q_k} = [A_0; A_1, \ldots, A_k]$$

for the k-th convergent and

$$\xi_k = [0; A_{k+1}, A_{k+2}, \ldots].$$

Next, we shall mention that there are lots of similarities between this continued fraction expansion and the continued fraction expansion of the real case but there are also some differences, too. We are going to state some properties that are straightforward to prove and that show similarity but divergency as well.

Lemma 1.4.1. *Let $\alpha \in \mathbb{F}_q((T^{-1}))$ be given. Then, we have*

1. $(P_k, Q_k) = 1$
2. $1 = |Q_0| < |Q_1| < |Q_2| < \dots$
3. $|Q_k| = \prod_{i=1}^{k} |A_i|$
4. $\left|\alpha - \frac{P_k}{Q_k}\right| = \frac{1}{|Q_k||Q_{k+1}|} < \frac{1}{|Q_k|^2}$
5. *Let $P, Q \in \mathbb{F}_q[T]$ with $Q \neq 0$ and*
$$\left|\alpha - \frac{P}{Q}\right| < \frac{1}{|Q|^2}.$$
Then, there is an integer $k \geq 0$ such that $P/Q = P_k/Q_k$.

Remark 1.4.1. Especially, from (4) and (5) of the last lemma, we obtain the following criterion for polynomials P and Q with $(P, Q) = 1$ to be denominator resp. enumerator of some convergent of $\alpha \in \mathbb{F}_q((T^{-1}))$

$$\exists k \geq 0 : P = P_k \text{ and } Q = Q_k \quad \Longleftrightarrow \quad \left|\alpha - \frac{P}{Q}\right| < \frac{1}{|Q|^2}$$

Note, that this is not true in the classical case.

1.4.2 Metric Theory in the Formal Laurent Series Case

Since we are interested in the metric theory, we consider the following subset of $\mathbb{F}_q((T^{-1}))$
$$H = \{\alpha \in \mathbb{F}_q((T^{-1}))||\alpha| < 1\}$$
which can be seen as an analogue of the interval $[0, 1]$. Observe, that H together with the restriction of the valuation on H is a compact abelian group. Let $\mathcal{H}$ denote its σ-Algebra of Borel sets. Then, it is well known that there exists an unique, translation invariant probability measure on the measure space $(H, \mathcal{H})$ that we are going to denote by h.

The probability measure h is quite easy to understand. Especially, the h-measure of balls is very easy to compute

Lemma 1.4.2. *Let $\alpha_0 \in H$ be given. Then, we have*

$$h\left(\{\alpha \in H||\alpha - \alpha_0| < q^{-k}\}\right) = q^{-k}$$

Proof. Follows immediatly from the translation invariance of h. $\square$

As a consequence of this lemma, we get a result that is easy and powerful at the same time

Theorem 1.4.1 (Lemma 2 in [20]). *Let $B_1, \dots, B_k$ be polynomials over $\mathbb{F}_q$ with $\deg B_i \geq 1$. Then, we have*

$$h\left(\{\alpha \in H | A_i(\alpha) = B_i, 1 \leq i \leq k\}\right) = q^{-2(\deg B_1 + \cdots \deg B_k)}.$$

Proof. Since the proof is nearly trivial, we repeat it. Just observe

$$\{\alpha \in H | A_i(\alpha) = B_i, 1 \leq i \leq k\} = \{\alpha \in H || \alpha - C/D| < 1/|D|^2\}$$

where $C/D = [0; B_1, \dots, B_k]$ and apply Lemma 1.4.2 along with (3) of Lemma 1.4.1. $\square$

This theorem implies the following result that has to be compared with the Gauss-Kuzmin Theorem in Section 1.2.1

Corollary 1.4.1. *We have*

$$h\left(\{\alpha \in H || \xi_k| < t\}\right) = q^{\lceil \log_q t \rceil}.$$

Proof. Observe

$$|\xi_k| = \frac{1}{|A_{k+1}|}$$

which is straightforward to prove. Now, the assertion follows by applying the last theorem. $\square$

Furthermore, Theorem 1.4.2 entails the following two consequences

Corollary 1.4.2. *Let $B \in \mathbb{F}_q[T]$ be given. Then, we have*

$$h\left(\{\alpha \in H | A_k(\alpha) = B\}\right) = q^{-2 \deg B}.$$

Corollary 1.4.3. *Let f be a function defined on the polynomials in $\mathbb{F}_q[T]$ with degree at least 1. Then, the sequence of random variables $(f(A_k))_{k \geq 1}$ on the probability space $(H, \mathcal{H}, h)$ is independent.*

The last two corollaries taken together imply that the sequence of random variables $(f(A_k))_{k \geq 1}$ is an independent and identically distributed one. Therefore, the metric theory of continued fractions in the Laurent series case is much easier than in the real case. Analogues of Theorem 1.2.2 and Theorem 1.2.3 as well can be obtained just by applying the theory of independent and identically distributed sequences of random variables. Furthermore, observe (compare with (1.22))

$$\sum_{k=1}^{n} \log_q |A_k| = \deg Q_n$$

by (2) of Lemma 1.4.1. Hence, we get the following Khintchine-Levy type theorem (see Theorem 1.2.5 in Section 1.2.1) in the Laurent series case

Theorem 1.4.2 (Corollary 3 in [20]). *For almost all $\alpha \in H$ we have*

$$\limsup_{n \longrightarrow \infty} \frac{|\deg Q_n - n \log_q \gamma|}{\sqrt{2\sigma^2 n \log\log n}} = 1,$$

where $\sigma = q/(q-1)^2$ and $\gamma = q^{q/(q-1)}$ is the Khintchine-Levy constant.

Therefore, we have obtained all results of the metric theory of continued fraction expansion stated in Section 1.2.1 in the Laurent series case as well. In Chapter 3, we will show that also the results in Section 1.2.2 posses analogues in the field of formal Laurent series.

Chapter 2

Metric Diophantine Approximation in the Real Case

As already mentioned in Section 1.3, in this work we are mainly interested in finding classes of functions for which the solution set of (1.23) satisfies strong probabilistic results such as the central limit theorem or the law of the iterated logarithm. Our investigations will yield improvements of all results known before in this direction (compare with Section 1.3.2). Especially, we are going to extend and solve LeVeque's second problem stated on page 22.

2.1 The Problem of LeVeque and its Solution

In Section 1.3.2, we mentioned LeVeque's problem raised in [18] that is the question whether a central limit theorem holds for the sequence of random variables $(T_n)_{n\geq 1}$ or not. In this section, we are going to extend and solve LeVeque's problem.

2.1.1 Statement of the Result

Suppose f is a positive real-valued function defined on the non-negative real numbers satisfying (1.25) and the following conditions

$$\textstyle\sum_{k=1}^{n} f(k)k^{-\delta} \ll \left(\sum_{k=1}^{n} f(k)\right)^{1/2}, \tag{2.1}$$

$$\textstyle\sum_{k=1}^{n} f(k)^2 \ll \left(\sum_{k=1}^{n} f(k)\right)^{1/2}, \tag{2.2}$$

where $0 < \delta < 1/2$. Futhermore, we put

$$F(n) = \sum_{k=1}^{n} \frac{f(\log k)}{k}.$$

Define a sequence of random variables as

$$X_n(x) := \#\{\langle p,q\rangle | 1 \leq q \leq n, q \equiv s\ (r),\\ p/q \text{ is a solution of (1.23)}\}. \tag{2.3}$$

Then, we have the following theorem that settles LeVeque's second problem

Theorem 2.1.1. *We have*

$$\lim_{n\longrightarrow\infty} \lambda\left[X_n \leq \sigma_1 F(n) + \omega\left(\sigma_2 F(n)\log F(n)\right)^{1/2}\right] = \Phi(\omega),$$

where

$$\sigma_1 = \frac{2}{r}, \quad \sigma_2 = \frac{12(s,r)\varphi(r)}{\pi^2 r C(s,r)},$$

and

$$C(s,r) = r^2 \prod_{p|r}\left(1-\frac{1}{p}\right)\prod_{p|\frac{r}{(s,r)}}\left(1+\frac{1}{p}\right).$$

Remark 2.1.1. Actually this solves an extension of LeVeque's second problem. First, observe that the class of functions considered by LeVeque is contained in our class since we assume weaker assumptions on f. Furthermore, LeVeque considered all solutions of (1.23) while we only take solutions into account that have denominators contained in an arithmetic progression.

Remark 2.1.2. Observe, that the assumptions on f are similar to the assumptions on f considered in Theorem 1.3.5. The main difference is assumption (2.2) which is actually only a technical one. We have introduced it in order to shorten the proof of our Theorem 2.1.1 but it could easily be dropped by combining our method with the proof method of Theorem 1.3.5 in Section 1.3.2.

The proof of Theorem 2.1.1 will take up most of this section and is performed in several steps. We are giving a short plan: in the next section, we will prepare the proof by approximating the sequence of random variables $(X_n)_{n\geq 1}$ several times. In Section 2.1.3, we are going to prove the corresponding central limit theorem for the approximating sequence of random variables and finally, Section 2.1.4 will be used to finish the proof of our main result by showing that the approximation is strong enough in order to entail asymptotic normality of $(X_n)_{n\geq 1}$ from the one of the approximating sequence (obtained in the section before).

2.1.2 Preliminaries

We start by approximating the sequence of random variables $(X_n)_{n\geq 1}$ several times. In the following, this will be a standard procedure that we are going to use throughout the work. First, we consider the following sequence of random variables

$$Y_k(x) := \#\{1 \le c \le a_{k+1} | cq_k \equiv s\ (r), c^2 \le \varphi_k f(\log cq_k)\}$$

that - by [31] - can be used to approximate $(X_n)_{n \ge 1}$

$$\sum_{q_{k+1} \le n} Y_k(x) \le X_n(x) \le \sum_{q_k \le n} Y_k(x) \tag{2.4}$$

As it was pointed out in [31], it is not possible to prove some mixing condition for the sequence Y_k and therefore, we have to approximate once more. In order to do this, we need to apply Theorem 1.2.5 of Section 1.2.1. Hence, we get that for each $\epsilon > 0$ there exists κ large enough such that

$$k \log \gamma - \kappa k^{1-\delta} \le \log q_k \le k \log \gamma + \kappa k^{1-\delta}, \quad k \ge 1 \tag{2.5}$$

for a subset F of $[0,1]$ with $\lambda(F) \ge 1 - \epsilon$. This yields

$$\begin{aligned} f((k+1)\log\gamma + \kappa(k+1)^{1-\delta}) \le f(\log q_{k+1}) \le f(\log cq_k) \\ \le f(\log q_k) \le f(k \log \gamma - \kappa k^{1-\delta}) \end{aligned}$$

for $x \in F$ and $1 \le c \le a_{k+1}$. Now, we set

$$f_1(k) := f((k+1)\log\gamma + \kappa(k+1)^{1-\delta}), \quad f_2(k) := f((k\log\gamma - \kappa k^{1-\delta}),$$

and define the following random variables

$$Z_k^{(i)}(x) := \#\{1 \le c | cq_k \equiv s\ (r), c^2 \le (a_{k+1} + 2\delta_{2,i}) f_i(k)\} \quad i = 1, 2,$$

where $\delta_{i,j}$ is the Kronecker function. It is easy to see that, we have

$$Z_k^{(1)}(x) \le Y_k(x) \le Z_k^{(2)}(x) \tag{2.6}$$

for all $x \in F$.

In the following, we focus on proving the central limit theorem for the sequences $(Z_k^{(i)})_{k \ge 0}$. By a theorem of Szüsz (see [30]), it is easy to see that these sequences are $\star$-mixing. However, central limit theorems for $\star$-mixing sequences of random variables cannot be applied because the variance of $(Z_k^{(i)})_{k \ge 0}$ is infinite. We use truncation in order to approximate the sequences $(Z_k^{(i)})_{k \ge 0}$ by a double sequence of random variables.

We put

$$F_i(n) := \sum_{k=1}^{n} f_i(k), \quad i = 1, 2,$$

and define the following double sequence of random variables

$$Z_{k,n}^{(i)}(x) := \#\{1 \le c \le \phi_n | cq_k \equiv s\ (r), c^2 \le (a_{k+1} + 2\delta_{2,i}) f_i(k)\}, \quad i = 1, 2,$$

where $\phi_n = [(F_i(n))^{1/2}(\log F_i(n))^{1/2-\rho}]$.

In order to compute moments of this sequences of random variables, we will apply Theorem 1.2.6 in Section 1.2.2. Furthermore, we need the following identity observed by Philipp in [23]

Lemma 2.1.1. *Set*

$$K(d) := \frac{r\varphi((d,r))}{C(r)(d,r)},$$

where $C(r)$ is defined in Theorem 1.2.6 in Section 1.2.2. Then, we have

$$\sum_{0\leq d<r} \sum_{\substack{1 \leq c \\ cd \equiv s(r)}} K(d)\frac{1}{c^2} = \frac{\pi^2}{6r}.$$

Finally, we need two technical lemmas. The first can be easily proved using ideas of [18] (compare with Lemma 4 there).

Lemma 2.1.2. *We have, as $n \longrightarrow \infty$,*

$$\sum_{k=1}^{n} f_i(k)^2 \ll F_i(n)^{1/2}, \quad i = 1, 2.$$

Lemma 2.1.3. *We have*

$$\sum_{0\leq d<r,(d,r)|s} (d,r)\varphi((d,r)) = (s,r)\varphi(r) \prod_{p|r,p\nmid\frac{r}{(s,r)}} \left(1 + \frac{1}{p}\right). \tag{2.7}$$

Proof. We start by observing that

$$\sum_{0\leq d<r,(d,r)|s} (d,r)\varphi((d,r)) = \sum_{k|(s,r)} k\varphi(k) \sum_{0\leq d<r,(d,r)=k} 1 = \sum_{k|(s,r)} k\varphi(k)\varphi\left(\frac{r}{k}\right).$$

Next, we write

$$r = \prod_{i=1}^{n} p_i^{e_i}, \quad (s,r) = \prod_{i=1}^{n} p_i^{f_i}$$

for the prime number decomposition of r resp. (s,r). We prove the claimed result by induction on n.

An easy calculation gives the case $n = 1$. Therefore, suppose that (2.7) is proved for all integers with at most $n-1$ prime numbers in the prime number decomposition. Denote by

$$\bar{r} = \prod_{i=1}^{n-1} p_i^{e_i} = \frac{r}{p_n^{f_n}}$$

and notice that

$$\sum_{k|(s,r)} k\varphi(k)\varphi\left(\frac{r}{k}\right) = \sum_{l=0}^{f_n}\sum_{k|(s,\bar r)} kp_n^l\varphi(kp_n^l)\varphi\left(\frac{\bar r}{k}p_n^{e_n-l}\right)$$
$$= \left(\sum_{k|(s,\bar r)} k\varphi(k)\varphi\left(\frac{\bar r}{k}\right)\right)\left(\sum_{k|p_n^{f_n}} k\varphi(k)\varphi\left(\frac{p_n^{e_n}}{k}\right)\right).$$

Applying the induction hypothesis twice yields

$$\sum_{0\le d<r,(d,r)|s} (d,r)\varphi((d,r)) = (s,\bar r)\varphi(\bar r)\prod_{p|\bar r, p\nmid\frac{\bar r}{(s,\bar r)}}\left(1+\frac{1}{p}\right)\cdot$$
$$\cdot\, p_n^{f_n}\varphi(p_n^{e_n})\prod_{p_n\nmid p_n^{e_n-f_n}}\left(1+\frac{1}{p}\right)$$
$$= (s,r)\varphi(r)\prod_{p|r,p\nmid\frac{r}{(s,r)}}\left(1+\frac{1}{p}\right)$$

which is the desired result. □

Using the last lemmas, we get for the meanvalue and the variance of the double sequence $Z_{k,n}^{(i)}$

Lemma 2.1.4. *For the random variables $Z_{k,n}^{(i)}$ introduced above we have*

$$\mu_{i,n} := \mathbf{E}\sum_{k\le n} Z_{k,n}^{(i)} = \frac{\pi^2}{6r\log 2}F_i(n) + O\left(F_i(n)^{1/2}\right) \tag{2.8}$$

and

$$\tau_{i,n}^2 := \mathbf{V}\sum_{k\le n} Z_{k,n}^{(i)} \sim \sigma F_i(n)\log F_i(n) \tag{2.9}$$

with

$$\sigma = \frac{(s,r)\varphi(r)}{rC(s,r)\log 2}.$$

where $C(s,r)$ is defined as in Theorem 2.1.1.

Proof. First, we observe

$$Z_{k,n}^{(i)}(x) = \sum_{0\le d<r}\sum_{\substack{1\le c\le\phi_n\\ cd\equiv s\ (r)}} \xi_k^{(d,c,i)}(x), \tag{2.10}$$

where

$$\xi_k^{(d,c,i)}(x) = \begin{cases} 1 & \text{if } q_k\equiv d\ (r) \text{ and } c^2\le(a_{k+1}+2\delta_{i,2})f_i(k),\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, we have

$$\mathbf{E}Z_{k,n}^{(i)} = \sum_{0\le d<r} \sum_{\substack{1\le c\le \phi_n\\ cd\equiv s\ (r)}} \mathbf{E}\xi_k^{(d,c,i)}. \tag{2.11}$$

Next notice that

$$c^2 \le (a_{k+1}+2\delta_{i,2})f_i(k) \Longleftrightarrow \xi_k \le 1/\left(\left\lceil \frac{c^2-2\delta_{i,2}f_i(k)}{f_i(k)}\right\rceil\right)$$

and by applying Theorem 1.2.6 in Section 1.2.2, we get

$$\mathbf{E}\xi_k^{(d,c,i)} = \frac{K(d)}{\log 2}\left(\frac{f_i(k)}{c^2} + \mathrm{O}\left(\frac{f_i(k)^2}{c^2}\right)\right)(1+\mathrm{O}(q^k)). \tag{2.12}$$

Hence, by combining (2.11) and (2.12),

$$\mathbf{E}Z_{k,n}^{(i)} = \frac{1}{\log 2}(1+\mathrm{O}(q^k))\Bigg(f_i(k)\sum_{0\le d<r}\sum_{\substack{1\le c\\ cd\equiv s\ (r)}}\frac{K(d)}{c^2} -$$

$$- f_i(k)\sum_{0\le d<r}\sum_{\substack{\phi_n<c\\ cd\equiv s\ (r)}}\frac{K(d)}{c^2} + f_i(k)^2\sum_{0\le d<r}\sum_{\substack{1\le c\le\phi_n\\ cd\equiv s\ (r)}}\mathrm{O}\left(\frac{K(d)}{c^2}\right)\Bigg).$$

Using Lemma 2.1.1 for the first double sum and simple estimates for the second and third gives

$$\mathbf{E}Z_{k,n}^{(i)} = \frac{1}{\log 2}\left(\frac{\pi^2}{6r}f_i(k) + \mathrm{O}\left(\frac{f_i(k)}{\phi_n}\right) + \mathrm{O}(f_i(k)^2)\right)(1+\mathrm{O}(q^k)). \tag{2.13}$$

Finally summing up and applying Lemma 2.1.2 implies (2.8).

In order to compute the variance, we need the second moment of $Z_{k,n}^{(i)}$. By (2.10), its not hard to see that

$$\mathbf{E}(Z_{k,n}^{(i)})^2 = \sum_{0\le d<r}\sum_{\substack{1\le c\le\phi_n\\ cd\equiv s\ (r)}} k_{c,d}\mathbf{E}\xi_k^{(d,l,i)}, \tag{2.14}$$

where $k_{c,d}$ is the number of pairs $\langle k_1,k_2\rangle$ of solutions of $xd\equiv s\ (r)$ with $k_1\le c, k_2\le c$ and either $k_1=c$ or $k_2=c$. Furthermore, we have

$$k_{c,d} = \begin{cases} 2\frac{(d,r)}{r}c + c_d & \text{if } (d,r)|s,\\ 0 & \text{otherwise,}\end{cases} \tag{2.15}$$

with a suitable constand c_d. Combining (2.12), (2.14), and (2.15) implies

$$\mathbf{E}(Z_{k,n}^{(i)})^2 = \frac{1}{\log 2}(1+\mathrm{O}(q^k)) \sum_{\substack{0 \le d < r \\ (d,r)|s}} \sum_{\substack{1 \le c \le \phi_n \\ cd \equiv s \ (r)}} K(d) \cdot$$

$$\cdot \left(2\frac{(d,r)}{r}c + c_d\right)\left(\frac{f_i(k)}{c^2} + \mathrm{O}\left(\frac{f_i(k)^2}{l^2}\right)\right).$$

We extend the last product and break the double sum into four parts. For one part we have

$$\sum_{\substack{0 \le d < r \\ (d,r)|s}} \sum_{\substack{1 \le c \le \phi_n \\ cd \equiv s \ (r)}} 2cK(d)\frac{(d,r)}{r}\frac{f_i(k)}{c^2}$$

$$= \frac{2f_i(k)}{C(r)} \sum_{\substack{0 \le d < r \\ (d,r)|s}} \varphi(d,r) \sum_{\substack{1 \le c \le \phi_n \\ cd \equiv s \ (r)}} \frac{1}{c}$$

$$= f_i(k)\sigma \log 2 \log F_i(n) + \mathrm{O}(f_i(k) \log\log F_i(n))$$

where Lemma 2.1.3 was used. It is easy to see that the other parts are bounded by either $f_i(k)$, $f_i(k)^2$ or $f_i(k)^2 \log F_i(n)$. Hence

$$\sum_{k \le n} \mathbf{E}(Z_{k,n}^{(i)})^2 = \sigma F_i(n) \log F_i(n) + \mathrm{O}(F_i(n) \log\log F_i(n))$$

$$+ \mathrm{O}(F_i(n)) + \mathrm{O}\left(\sum_{k=1}^{n} f_i(k)^2\right) + \mathrm{O}\left(\log F_i(n) \sum_{k=1}^{n} f_i(k)^2\right),$$

and taking Lemma 2.1.2 into account implies

$$\sum_{k \le n} \mathbf{E}(Z_{k,n}^{(i)})^2 \sim \sigma F_i(n) \log F_i(n). \tag{2.16}$$

Next, observe by (2.13) and Lemma 2.1.2 that

$$\sum_{k \le n} (\mathbf{E}Z_{k,n}^{(i)})^2 \ll (F_i(n))^{1/2}, \tag{2.17}$$

and we finish the proof by considering the covariance of $Z_{k_1,n}^{(i)}$ and $Z_{k_2,n}^{(i)}$ with $k_1 \le k_2$.

By [30], it is plain that $(Z_{k,n}^{(i)})_{k \le n}$ is $\star$-mixing, whereas ϕ in (1.4) is equal to q^n with q from Theorem 1.2.6 in Section 1.2.2. Hence, by using Lemma 1.1.1 in Section 1.1.2

$$\begin{aligned} |\mathbf{E}Z_{k_1,n}^{(i)} Z_{k_2,n}^{(i)} - \mathbf{E}Z_{k_1}^{(i)}\mathbf{E}Z_{k_2}^{(i)}| &\ll q^{k_2-k_1}\mathbf{E}Z_{k_1,n}^{(i)}\mathbf{E}Z_{k_2,n}^{(i)} \\ &\ll q^{k_2-k_1} f_i(k_1) f_i(k_2), \end{aligned} \tag{2.18}$$

where the last estimate follows from (2.13). Combining (2.16), (2.17), and (2.18) immediatly gives (2.9). □

The next step is normalization. Therefore, we define

$$\eta_{k,n}^{(i)} := (Z_{k,n}^{(i)} - \mathbf{E}Z_{k,n}^{(i)})/\tau_{i,n}.$$

Our main goal is to prove the central limit theorem for the double sequence $\eta_{k,n}^{(i)}$. In order to do this, we collect a few useful properties.

Lemma 2.1.5. *The double sequence $\eta_{k,n}^{(i)}$ satisfies the following properties*

1. *$\eta_{k,n}^{(i)}$ is uniformly strong mixing (this is nothing else than strong mixing but uniformly over the rows, see [21] for a precise definition),*

2. $$\mathbf{V}\sum_{k\leq n}\eta_{k,n}^{(i)} = 1,$$

3. $$|\eta_{k,n}^{(i)}| \leq \epsilon_n \quad \textit{and} \quad \epsilon_n \longrightarrow 0 \textit{ as } n \longrightarrow \infty,$$

4. $$\mathbf{V}\sum_{k\in I}\eta_{k,n}^{(i)} = \sum_{k\in I}\mathbf{V}\eta_{k,n}^{(i)} + O\left(\psi_1(n)\right),$$

   where $I \subseteq \{0,1,\ldots,n\}$, the implied constant doesn't depend on I, and $\psi_1(n) \longrightarrow 0$ as $n \longrightarrow \infty$,

5. $$\mathbf{E}\left(\sum_{k\in I}\eta_{k,n}^{(i)}\right)^4 \ll \sum_{k\in I}\mathbf{E}(Z_{k,n}^{(i)})^4 + \left(\sum_{k\in I}\mathbf{E}(Z_{k,n}^{(i)})^2\right)^2 + O(\psi_2(n)),$$

   where $I \subseteq \{0,1,\ldots,n\}$, the implied constant doesn't depend on I, and $\psi_2(n) \longrightarrow 0$ as $n \longrightarrow \infty$.

Remark 2.1.3. Inequality (5) is - except of the error term - an analogue of the famous Rosenthal inequality in the independent case.

Proof. (1) is an easy consequence of a theorem due to Szüsz (see [30]) and (2) follows immediatly from the definition of $\eta_{k,n}^{(i)}$. Furthermore by (2.9), (2.13), and the definition of $Z_{k,n}^{(i)}$ property (3) is obvious.

In order to prove (4) observe

$$\mathbf{V}\sum_{k\in I}Z_{k,n}^{(i)} = \sum_{k\in I}\mathbf{V}Z_{k,n}^{(i)} + 2\sum_{\substack{0\leq k_1<k_2\leq n\\ k_1,k_2\in I}}(\mathbf{E}Z_{k_1,n}^{(i)}Z_{k_2,n}^{(i)} - \mathbf{E}Z_{k_1,n}^{(i)}\mathbf{E}Z_{k_2,n}^{(i)}).$$

By Lemma 2.1.2 and (2.18), we have

$$2 \sum_{\substack{0 \le k_1 < k_2 \le n \\ k_1, k_2 \in I}} (\mathbf{E} Z_{k_1,n}^{(i)} Z_{k_2,n}^{(i)} - \mathbf{E} Z_{k_1,n}^{(i)} \mathbf{E} Z_{k_2,n}^{(i)}) \ll \sum_{k=1}^{n} f_i(k)^2 \ll F_i(n)^{1/2}$$

and hence (4) follows.

We are left with the proof of property (5). Therefore, we expand the left hand side of (5) by the multinomial theorem

$$\mathbf{E} \left(\sum_{k \in I} \eta_{k,n}^{(i)} \right)^4 = \sum_{e_1 + \ldots + e_t = 4} \binom{4}{e_1, \ldots, e_t} \mathbf{E} (\eta_{k_1,n}^{(i)})^{e_1} \cdots (\eta_{k_t,n}^{(i)})^{e_t}, \tag{2.19}$$

where $I = \{k_1, \ldots, k_t\}$. We split the sum into several parts according to the powers on the right hand side.

The part with the 4th-powers immediatly gives the first sum on the right hand side of (5).

Next, we consider the part where two 2nd-powers occur and take the mixing property of $\eta_{k,n}^{(i)}$ into account

$$\sum_{\substack{0 \le l_1 < l_2 \le n \\ l_1, l_2 \in I}} \binom{4}{2,2} \mathbf{E} (\eta_{l_1,n}^{(i)})^2 (\eta_{l_2,n}^{(i)})^2 =$$

$$\sum_{\substack{0 \le l_1 < l_2 \le n \\ l_1, l_2 \in I}} \binom{4}{2,2} \mathbf{E} (\eta_{l_1,n}^{(i)})^2 \mathbf{E} (\eta_{l_2,n}^{(i)})^2 (1 + \mathrm{O}(q^{l_2 - l_1})) \ll \left(\sum_{k \in I} \mathbf{E} (\eta_{k,n}^{(i)})^2 \right)^2 .$$

Hence, we have the second sum on the right hand side of (5).

In order to finish the proof, we have to show that all other parts are bounded by a function which tends to 0 as n tends to infinity. We use ideas of [25]. First consider

$$\sum_{\substack{0 \le l_1 < l_2 < l_3 < l_4 \le n \\ l_1, l_2, l_3, l_4 \in I}} \mathbf{E} \eta_{l_1,n}^{(i)} \eta_{l_2,n}^{(i)} \eta_{l_3,n}^{(i)} \eta_{l_4,n}^{(i)}.$$

We break the sum into two parts $\sum = \sum^* + \sum^{**}$ according to whether $l_2 - l_1 \le [F_i(n)^{1/2}]$ and $l_3 - l_2 \le [F_i(n)^{1/2}]$ or not. The first part can be estimated as follows

$$\begin{aligned} \sum\nolimits^* \mathbf{E} \eta_{l_1,n}^{(i)} \eta_{l_2,n}^{(i)} \eta_{l_3,n}^{(i)} \eta_{l_4,n}^{(i)} &\ll \sum\nolimits^* \mathbf{E} |\eta_{l_1,n}^{(i)} \eta_{l_2,n}^{(i)} \eta_{l_3,n}^{(i)}| \mathbf{E} |\eta_{l_4,n}^{(i)}| q^{l_4 - l_3} \\ &\ll \sum\nolimits^* \mathbf{E} |\eta_{l_1,n}^{(i)}| \mathbf{E} |\eta_{l_2,n}^{(i)}| \mathbf{E} |\eta_{l_3,n}^{(i)}| \mathbf{E} |\eta_{l_4,n}^{(i)}| q^{l_4 - l_3} \\ &\ll \frac{1}{\tau_{i,n}^4} \sum\nolimits^* \mathbf{E} Z_{l_1,n}^{(i)} \mathbf{E} Z_{l_2,n}^{(i)} \mathbf{E} Z_{l_3,n}^{(i)} \mathbf{E} Z_{l_4,n}^{(i)} q^{l_4 - l_3}. \end{aligned}$$

Using (2.9) and (2.13), we get

$$\begin{aligned}\sum\nolimits^{*} \mathbf{E}\eta^{(i)}_{l_1,n}\eta^{(i)}_{l_2,n}\eta^{(i)}_{l_3,n}\eta^{(i)}_{l_4,n} &\ll \frac{F_i(n)}{(F_i(n)\log F_i(n))^2}\sum_{k=1}^{n} f_i(k)\\ &= \frac{1}{(\log F_i(n))^2}.\end{aligned}$$

In order to estimate the second part, we break the sum again into two parts $\sum^{**} = \sum' + \sum''$ according to which of the two conditions is violated. We consider the first part

$$\begin{aligned}\sum\nolimits' \mathbf{E}\eta^{(i)}_{l_1,n}\eta^{(i)}_{l_2,n}\eta^{(i)}_{l_3,n}\eta^{(i)}_{l_4,n} &\ll \sum\nolimits' \mathbf{E}|\eta^{(i)}_{l_1,n}|\mathbf{E}|\eta^{(i)}_{l_2,n}\eta^{(i)}_{l_3,n}\eta^{(i)}_{l_4,n}|q^{l_2-l_1}\\ &\ll \frac{q^{(F_i(n))^{1/2}}}{\tau^4_{i,n}}\sum\nolimits' \mathbf{E}Z^{(i)}_{l_1,n}\mathbf{E}Z^{(i)}_{l_2,n}\mathbf{E}Z^{(i)}_{l_3,n}\mathbf{E}Z^{(i)}_{l_4,n}\\ &\ll \frac{q^{(F_i(n))^{1/2}}}{\tau^4_{i,n}}\left(\sum_{k\le n}\mathbf{E}Z^{(i)}_{k,n}\right)^4.\end{aligned}$$

Using (2.8) and (2.9), this implies

$$\begin{aligned}\sum\nolimits' \mathbf{E}\eta^{(i)}_{l_1,n}\eta^{(i)}_{l_2,n}\eta^{(i)}_{l_3,n}\eta^{(i)}_{l_4,n} &\ll \frac{F_i(n)^4 q^{(F_i(n))^{1/2}}}{(F_i(n)\log F_i(n))^2}\\ &= \frac{F_i(n)^2 q^{(F_i(n))^{1/2}}}{(\log F_i(n))^2},\end{aligned}$$

which tends to 0 as n tends to ∞. The second part is treated similarly

$$\begin{aligned}\sum\nolimits'' \mathbf{E}\eta^{(i)}_{l_1,n}\eta^{(i)}_{l_2,n}\eta^{(i)}_{l_3,n}\eta^{(i)}_{l_4,n} \ll &\sum\nolimits'' \mathbf{E}\eta^{(i)}_{l_1,n}\eta^{(i)}_{l_2,n}\mathbf{E}\eta^{(i)}_{l_3,n}\eta^{(i)}_{l_4,n}\\ &+ \sum\nolimits'' \mathbf{E}|\eta^{(i)}_{l_1,n}\eta^{(i)}_{l_2,n}|\mathbf{E}|\eta^{(i)}_{l_3,n}\eta^{(i)}_{l_4,n}|q^{l_3-l_2}\\ \ll &\sum\nolimits'' \mathbf{E}|\eta^{(i)}_{l_1,n}|\mathbf{E}|\eta^{(i)}_{l_1,n}|q^{l_2-l_1}\mathbf{E}|\eta^{(i)}_{l_3,n}|\mathbf{E}|\eta^{(i)}_{l_4,n}|q^{l_4-l_3}\\ &+ q^{(F_i(n))^{1/2}}\sum\nolimits'' \mathbf{E}|\eta^{(i)}_{l_1,n}|\mathbf{E}|\eta^{(i)}_{l_2,n}|\mathbf{E}|\eta^{(i)}_{l_3,n}|\mathbf{E}|\eta^{(i)}_{l_4,n}|\\ \ll &\frac{1}{\tau^4_{i,n}}\left(\sum_{k\le n}\mathbf{E}Z^{(i)}_{k,n}\right)^2 + \frac{q^{(F_i(n))^{1/2}}}{\tau^4_{i,n}}\left(\sum_{k\le n}\mathbf{E}Z^{(i)}_{k,n}\right)^4.\end{aligned}$$

Using (2.8) and (2.9) once more, we get

$$\sum\nolimits'' \mathbf{E}\eta^{(i)}_{l_1,n}\eta^{(i)}_{l_2,n}\eta^{(i)}_{l_3,n}\eta^{(i)}_{l_4,n} \ll \frac{1}{(\log F_i(n))^2} + \frac{F_i(n)^2 q^{(F_i(n))^{1/2}}}{(\log F_i(n))^2},$$

which again tends to 0 as n tends to ∞.

All remaining parts of the right hand side of (2.19), namely $\sum_{l_1<l_2}\mathbf{E}(\eta^{(i)}_{l_1,n})^3\eta^{(i)}_{l_2,n}$, $\sum_{l_1<l_2}\mathbf{E}\eta^{(i)}_{l_1,n}(\eta^{(i)}_{l_2,n})^3$, $\sum_{l_1<l_2<l_3}\mathbf{E}(\eta^{(i)}_{l_1,n})^2\eta^{(i)}_{l_2,n}\eta^{(i)}_{l_3,n}$, $\sum_{l_1<l_2<l_3}\mathbf{E}\eta^{(i)}_{l_1,n}(\eta^{(i)}_{l_2,n})^2\eta^{(i)}_{l_3,n}$, and $\sum_{l_1<l_2<l_3}\mathbf{E}\eta^{(i)}_{l_1,n}\eta^{(i)}_{l_2,n}(\eta^{(i)}_{l_3,n})^2$, can be treated in the same manner. $\square$

We shall use the properties of the last lemma together with standard techniques in proving central limit theorems for weakly dependent random variables and some ideas of [21] in order to prove asymptotic normality of $\eta^{(i)}_{k,n}$ in the next section.

2.1.3 Central Limit Theorem for the Double Sequence $\eta^{(i)}_{k,n}$

Througout the section, we will suppress the dependence on i. The next goal is the proof of the central limit theorem for $\eta_{k,n}$. We proceed in several steps.

Step 1: Blocking.

Because of $\epsilon_n \longrightarrow 0$, $q^n \longrightarrow 0$, and $\psi_i(n) \longrightarrow 0, i = 1,2$ as $n \longrightarrow \infty$ there exist a sequence c_n with the following properties, as $n \longrightarrow \infty$,

$$
\begin{aligned}
c_n &\longrightarrow \infty, && (2.20)\\
c_n\epsilon_n &\longrightarrow 0, && (2.21)\\
c_n q^{[\epsilon_n^{-1}]} &\longrightarrow 0, && (2.22)\\
c_n\psi_1(n) &\longrightarrow 0, && (2.23)\\
c_n\psi_2(n) &\longrightarrow 0. && (2.24)
\end{aligned}
$$

We fix n and define a sequence of integers by

$$m_{0,n} := 0,$$

and for $l = 0,1,2,\dots$ by

$$
\begin{aligned}
m_{2l+1,n} &:= \min\{m > m_{2l} | \sum_{k=m_{2l}+1}^{m} \mathbf{V}\eta_{k,n} \geq c_n^{-1}\}, && (2.25)\\
m_{2l+2,n} &:= m_{2l+1} + [\epsilon_n^{-1}]. && (2.26)
\end{aligned}
$$

There are two possibilities how this inductive procedure can stop. On the one hand, if we have constructed (2.26) and there are no random variables for (2.25) left or if the sum of the variances of the remaining random variables is too small to be at least c_n^{-1}. In the first case, we put $m_{2l+2} := n$ and in the second case, we increase m_{2l} by the number of remaining random variables.

Next, we define

$$
\begin{aligned}
I_{l,n} &= \{k | m_{2l} < k \leq m_{2l+1}\},\\
J_{l,n} &= \{k | m_{2l+1} < k \leq m_{2l+2}\},
\end{aligned}
$$

and finally

$$\begin{aligned}\xi_{l,n} &= \sum_{k\in I_{l,n}} \eta_{k,n},\\ \zeta_{l,n} &= \sum_{k\in J_{l,n}} \eta_{k,n},\end{aligned}$$

where $0 \le l < w_n$ and w_n is the number of $I_{l,n}$ (resp. $J_{l,n}$) obtained from the construction above.

We start with an easy observation. By Lemma 2.1.5 (4) and the definition of $I_{l,n}$, we have

$$c_n^{-1} w_n \le \sum_{l=0}^{w_n-1} \sum_{k\in I_{l,n}} \mathbf{V}\eta_{k,n} \le \sum_{k\le n} \mathbf{V}\eta_{k,n} \le K,$$

for a suitable constant K. Hence

$$w_n \le K c_n. \tag{2.27}$$

The next step is to show that it is enough to prove the central limit theorem for the double sequence $\xi_{l,n}$.

Step 2: $\zeta_{l,n}$ is negligible.

In order to show this, we consider the variance of $\zeta_{l,n}$. First observe, by Lemma 2.1.5 (3),(4), and the definition of $J_{w_n-1,n}$, that

$$\begin{aligned}\mathbf{V}\zeta_{w(n)-1,n} = \mathbf{V}\sum_{k\in J_{w_n-1,n}} \eta_{k,n} &= \sum_{k\in J_{w_n-1,n}} \mathbf{V}\eta_{k,n} + \mathrm{O}(\psi_1(n))\\ &\le \epsilon_n^{-1} \max_{k\le n} \mathbf{E}\eta_{k,n}^2 + c_n^{-1} + \mathrm{O}(\psi_1(n))\\ &= \epsilon_n + c_n^{-1} + \mathrm{O}(\psi_1(n)),\end{aligned} \tag{2.28}$$

which tends to 0 as n tends to ∞. Taking once more Lemma 2.1.5 (3),(4), and the definition of $J_{l,n}$ together with (2.27) into account, we get

$$\begin{aligned}\mathbf{V}\sum_{l<w(n)-1} \zeta_{l,n} &= \sum_{l<w(n)-1} \sum_{k\in J_{l,n}} \mathbf{V}\eta_{l,n} + \mathrm{O}(\psi_1(n))\\ &\le w(n)\epsilon_n^{-1} \max_{k\le n} \mathbf{E}\eta_{k,n}^2 + \mathrm{O}(\psi_1(n))\\ &\ll c_n\epsilon_n + \mathrm{O}(\psi_1(n)),\end{aligned} \tag{2.29}$$

which tends to 0 as n tends to ∞, too (here, (2.21 was used).

Next, observe

$$\mathbf{V}\sum_{l<w_n} \zeta_{l,n} = \mathbf{V}\zeta_{w(n)-1,n} + \mathbf{V}\sum_{l<w(n)-1} \zeta_{l,n} + \mathrm{O}(\psi_1(n)),$$

which together with (2.28) and (2.29) implies

$$\mathbf{V} \sum_{l<w_n} \zeta_{l,n} \to 0, \quad \text{as } n \to \infty. \tag{2.30}$$

Therefore, as $n \to \infty$,

$$\lambda \left[\sum_{l<w_n} \zeta_{l,n} \geq \epsilon \right] \leq \left(\mathbf{V} \sum_{l<w_n} \zeta_{l,n} \right) / \epsilon^2 \to 0,$$

and the second assertion is proved.

It follows that asymptotic normality of the double sequence $\xi_{l,n}$ is sufficient to imply the central limit theorem for $\eta_{l,n}$. The next step is a standard approximation with independent random variables.

Step 3: Approximation by Independent Random Variables.

First its plain by Lemma 2.1.5 (4) and (2.30) that

$$\mathbf{V} \sum_{l<w_n} \xi_{l,n} \longrightarrow 1, \quad \text{as } n \longrightarrow \infty. \tag{2.31}$$

Next, we consider

$$\sum_{l<w_n} \mathbf{V}\xi_{l,n} = \mathbf{V} \sum_{l<w_n} \xi_{l,n} + \mathrm{O}(c_n \psi_1(n)), \tag{2.32}$$

where (2.27) and again Lemma 2.1.5 (4) was used. Hence, by combining (2.31) and (2.32) and taking (2.23) into account, we have

$$a_n := \sum_{l<w_n} \mathbf{V}\xi_{l,n} \longrightarrow 1, \quad \text{as } n \longrightarrow \infty.$$

By a standard argument (see for instance [21] p712) and using (2.21) one can approximate the double sequence $\xi_{l,n}$ by an independent double sequence $\bar{\xi}_{l,n}$ with the same distribution. Therefore, we are left with proving asymptotic normality of $\bar{\xi}_{l,n}$.

Step 4: Asymptotic Normality of $\bar{\xi}_{l,n}$.

First, we normalize

$$\tilde{\xi}_{l,n} := \bar{\xi}_{l,n} / a_n.$$

Notice that it is sufficient to prove Lyapunov's condition for $\tilde{\xi}_{l,n}$, namely

$$\sum_{l<w_n} \mathbf{E}\tilde{\xi}_{l,n}^{2+\tau} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,$$

where $\tau > 0$ is a real constant. Therefore, observe

$$\sum_{k \in I_l} \mathbf{V}\eta_{k,n} \leq c_n^{-1} + \max_{k \leq n} \mathbf{E}\eta_{k,n}^2 \leq c_n^{-1} + \epsilon_n^2, \tag{2.33}$$

by the definition of I_l and Lemma 2.1.5 (3). Next, we have

$$\mathbf{E}\eta_{k,n}^4 \leq \epsilon_n^2 \mathbf{E}\eta_{k,n}^2, \tag{2.34}$$

where again Lemma 2.1.5 (3) was used. Hence, Lemma 2.1.5 (4), (2.33), (2.34) implies

$$\begin{aligned} \mathbf{E}\xi_{l,n}^4 &= \mathbf{E}\left(\sum_{k \in I_l} \eta_{k,n}\right)^4 \\ &\ll \sum_{k \in I_l} \mathbf{E}\eta_{k,n}^4 + \left(\sum_{k \in I_l} \mathbf{E}\eta_{k,n}^2\right)^2 + \mathrm{O}(\psi_2(n)) \\ &\ll (\epsilon_n + c_n^{-1})^2 + \mathrm{O}(\psi_2(n)). \end{aligned}$$

Furthermore, notice

$$\sum_{l < w_n} \mathbf{E}\tilde{\xi}_{l,n}^4 = \frac{1}{a_n^4} \sum_{l < w_n} \mathbf{E}\xi_{l,n}^4 \ll w_n(\epsilon_n + c_n^{-1})^2 + \mathrm{O}(w_n \psi_2(n)),$$

and together with (2.23), (2.24), (2.27), we get

$$\sum_{l < w_n} \mathbf{E}\tilde{\xi}_{l,n}^4 \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,$$

which proves the asymptotic normality of $\tilde{\xi}_{l,n}$

$$\sum_{l < w_n} \tilde{\xi}_{l,n} \rightharpoonup \mathcal{N}(0,1), \quad \text{as } n \to \infty.$$

Since $a_n \longrightarrow 1$, as $n \longrightarrow \infty$, and because of step 2 and step 3 we obtain

$$\sum_{k \leq n} \eta_{k,n} \rightharpoonup \mathcal{N}(0,1), \quad \text{as } n \to \infty,$$

so that the proof of the central limit theorem for the double sequence $\eta_{k,n}$ is finished.

2.1.4 Proof of Theorem 2.1.1

We finish the proof by showing that the central limit theorem for $\eta_{k,n}^{(i)}$ implies the asymptotic normality of X_n (with suitable normalization). In order to do this, we need

Lemma 2.1.6. *We have*

$$\mathbf{E}\left|\sum_{k\le n} Z_k^{(i)} - \sum_{k\le n} Z_{k,n}^{(i)}\right| \ll \frac{(F_i(n))^{1/2}}{(\log F_i(n))^{1/2-\rho}}.$$

Proof. First observe

$$\begin{aligned}\mathbf{E}\left|\sum_{k\le n} Z_k^{(i)} - \sum_{k\le n} Z_{k,n}^{(i)}\right| &= \sum_{k\le n} \mathbf{E}(Z_k^{(i)} - Z_{k,n}^{(i)}) \\ &\le \sum_{k\le n} \mathbf{E}\#\{\phi_n < c | c^2 \le (a_{k+1} + 2\delta_{i,2}) f_i(k)\} \qquad (2.35) \\ &= \sum_{k\le n} \sum_{\phi_n < c} \mathbf{E}\xi_k^{(c,i)},\end{aligned}$$

where

$$\xi_k^{(c,i)}(x) := \begin{cases} 1 & \text{if } c^2 \le (a_{k+1} + 2\delta_{i,2}) f_i(k), \\ 0 & \text{otherwise.} \end{cases}$$

Using Theorem 1.2.6 of Section 1.2.2, we get

$$\mathbf{E}\xi_k^{(c,i)} \ll \frac{f_i(k)}{c^2},$$

and together with (2.35)

$$\mathbf{E}\left|\sum_{k\le n} Z_k^{(i)} - \sum_{k\le n} Z_{k,n}^{(i)}\right| \ll F_i(n) \sum_{\phi_n < c} \frac{1}{c^2} \ll \frac{F_i(n)}{F_i(n)^{1/2} (\log F_i(n))^{1/2-\rho}},$$

which proves the assertion. □

The above Lemma together with (2.9) implies

$$\begin{aligned}\lambda\left[\left(\sum_{k\le n} Z_k^{(i)} - \mathbf{E}Z_{k,n}^{(i)}\right)/\tau_{i,n} - \sum_{k\le n} \eta_{k,n}^{(i)} \ge \epsilon\right] \\ \le \left(\mathbf{E}\left|\sum_{k\le n} Z_k^{(i)} - \sum_{k\le n} Z_{k,n}^{(i)}\right|\right)/(\epsilon\tau_{i,n}) \\ \ll \frac{1}{(\log F_i(n))^{1-\rho}},\end{aligned}$$

and hence, as $n \longrightarrow \infty$,

$$\left(\sum_{k\le n} Z_k^{(i)} - \frac{\pi^2}{6r\log 2}F_i(n)\right)/(\sigma F_i(n)\log F_i(n))^{1/2} \rightharpoonup \mathcal{N}(0,1), \qquad (2.36)$$

where (2.8) and (2.9) was used.

Next, we need the following technical Lemma due to Philipp [23]

Lemma 2.1.7. *Let g_1 (resp. g_2) be the inverse function of $\gamma^{(k+1)}\exp(\kappa(k+1)^{1-\delta})$ (resp. $\gamma^k exp(-\kappa k^{1-\delta})$). Then we have*

$$F_i(g_i(n)) = \frac{1}{\log\gamma}F(n) + O(F(n)^{1/2}).$$

By (2.4), (2.45), (2.6), and the definition of g_i, we have

$$\sum_{k\le g_1(n)} Z_k^{(1)}(x) \le \sum_{k\le g_1(n)} Y_k(x) \le \sum_{q_{k+1}\le n} Y_k(x) \le X_n(x),$$

and

$$X_n(x) \le \sum_{q_k\le n} Y_k(x) \le \sum_{k\le g_2(n)} Y_k(x) \le \sum_{k\le g_2(n)} Z_k^{(2)}(x),$$

for $x \in F$.

A standard argument together with (2.36) and Lemma 2.1.7 yields the result.

2.1.5 Remarks

An interesting question is whether the law of the iterated logarithm holds or not. Theorem 2.1.1 suggests the following conjecture (for the notation see Section 2.1.1.)

Conjecture 2.1.1. *We have for almost all $x \in [0,1]$*

$$\limsup_{n\longrightarrow\infty} \frac{|X_n(x) - \sigma_1 F(n)|}{(2\sigma_2 F(n)\log F(n)\log\log F(n))^{1/2}} = 1,$$

where

$$\sigma_1 = \frac{2}{r}, \quad \sigma_2 = \frac{12(s,r)\varphi(r)}{\pi^2 r C(s,r)},$$

and

$$C(s,r) = r^2\prod_{p|r}\left(1-\frac{1}{p}\right)\prod_{p|\frac{r}{(s,r)}}\left(1+\frac{1}{p}\right).$$

This result would imply that Schmidt's Theorem [26] is nearly best possible, especially, the *log*-factor cannot be removed (notice that the factor $\Omega(h)^{1/2}$ in Schmidt's Theorem can be removed as was pointed out by Harman in [11]).

2.2 Invariance Principles

In this section, we give a common extension of Theorem 1.3.3 and Theorem 1.3.5 of Section 1.3.2 by proving almost sure and distribution type invariance principles. Our results entail several corollaries e.g. a functional central limit theorem and a Strassen's type version of the law of the iterated logarithm.

2.2.1 Statement of the Results

Let f be a positive, real-valued function defined on the non-negative real numbers satisfying conditions (1.25) and (1.26) (the conditions in Philipp's Theorem 1.3.5). We define a sequence of random variables as

$$\begin{aligned} X_n^{(d)}(x) := \#\{\langle p,q\rangle | 1 \leq q \leq n, q \equiv s\ (r), \\ (p,q) \leq d, p/q \text{ is a solution of (1.23)}\}, \end{aligned}$$

where $n \geq 0$ and $d \geq 1, r \geq 1, s$ are fixed integers.

Furthermore, we put

$$F(n) = \sum_{k=1}^{n} \frac{f(\log k)}{k},$$

and

$$\sigma = \sigma_d = \frac{12r}{C(r)\pi^2} \sum_{0 \leq q < r} \frac{\varphi((q,r))}{(q,r)} \sum_{\substack{1 \leq c \leq d \\ cq \equiv s\ (r)}} \frac{1}{c^2},$$

$$\tau^2 = \tau_d^2 = \frac{12r}{C(r)\pi^2} \sum_{0 \leq q < r} \frac{\varphi((q,r))}{(q,r)} \sum_{\substack{1 \leq c \leq d \\ cq \equiv s\ (r)}} \frac{k_{c,q}}{c^2},$$

where $k_{c,q}$ is the number of pairs $\langle k_1, k_2\rangle$ of solutions of $xq \equiv s\ (r)$ with $k_1 \leq c, k_2 \leq c$ and either $k_1 = c$ or $k_2 = c$.

Finally, we set

$$n_t = \begin{cases} \max\{n | \tau^2 F(n) \leq t\} & \text{if } t \geq \tau^2 F(1), \\ 0 & \text{otherwise,} \end{cases}$$

where $t \geq 0$.

Now, let $([0,1]^2, \mathcal{B}_2, \lambda_2)$ denote the unit square with Lebesgue measure. Using the above notation, we define on this probability space a stochastic process X by setting for $\langle x_1, x_2\rangle \in [0,1]^2$

$$X(t) = X(t; x_1, x_2) = X_{n_t}^{(d)}(x_1) - \sigma F(n_t)$$

(observe that the right hand side is independent of x_2). Adjoining a uniformly distributed random variable independent of the entire sequence $(X_n^{(d)}(x_1))_{n \geq 1}$ guarantees that the probability space is rich enough.

Theorem 2.2.1. *There exists a sequence* $(Y_i(x_1, x_2))_{i\geq 1}$ *of independent, standard normal* $\mathcal{N}(0,1)$ *random variables defined on the above probability space such that, as* $n \longrightarrow \infty$,

$$X(n) - \sum_{i\leq n} Y_i = o((n \log\log n)^{1/2}) \quad a.s. \tag{2.37}$$

and

$$\lambda_2 \left[\frac{1}{\sqrt{n}} \max_{k\leq n} \left| X(k) - \sum_{i\leq k} Y_i \right| \geq \epsilon \right] \longrightarrow 0 \tag{2.38}$$

for all $\epsilon > 0$.

If f satisfies slightly stronger assumptions, we can prove even more

Theorem 2.2.2. *Let* f *satisfy (1.25), (1.26) and the following additional condition*

$$\sum_{k=1}^{n} f^2(k) \ll \left(\sum_{k=1}^{n} f(k) \right)^{1-\delta_3} \tag{2.39}$$

where $0 < \delta_3 < 1$. *Then, there exists a sequence* $(Y_i(x_1, x_2))_{i\geq 1}$ *of independent, standard normal* $\mathcal{N}(0,1)$ *random variables defined on the above probability space such that, as* $n \longrightarrow \infty$,

$$X(n) - \sum_{i\leq n} Y_i \ll n^{1/2-\lambda} \quad a.s. \tag{2.40}$$

where $0 < \lambda < 1/2$ *is a real constant.*

Remark 2.2.1. Notice, that there can be given an equivalent formulation of Theorem 2.2.1 and Theorem 2.2.2 by using standard Brownian motion. For instance (2.40) can be also stated as follows

$$X(t) - W(t) \ll t^{1/2-\lambda} \tag{2.41}$$

where $W(t)$ denotes standard Brownian motion on the unit square.

In order to see that (2.40) and (2.41) are equivalent, we only have to show that (2.40) implies (2.41) since the other direction is obvious. Therefore, observe by the definition of n_t

$$\sigma F(n_t) = \frac{\sigma}{\tau^2} t + \mathrm{O}(1).$$

Consequently,

$$X([t]) \leq X(t) + \mathrm{O}(1) \leq X([t]+1) + \mathrm{O}(1)$$

for $t \geq \min\{\tau^2 F(1), 1\}$. This implies

$$\Big|X(t) - \sum_{i \leq t} Y_i\Big| \leq \max\Big\{\Big|X([t]) - \sum_{i \leq [t]} Y_i\Big|, \\ \Big|X([t]+1) - \sum_{i \leq [t]+1} Y_i\Big| + |Y_{[t]+1}|\Big\} + \mathrm{O}(1).$$

Furthermore, it is plain by the Borel Cantelli Lemma that

$$Y_i \leq i^{1/2-\epsilon} \quad \text{a.s.}$$

for all $0 < \epsilon < 1/2$ and therefore, by using (2.40)

$$X(t) - \sum_{i \leq t} Y_i \ll t^{1/2-\lambda} \quad \text{a.s.} \tag{2.42}$$

for $t \geq \min\{\tau^2 F(1), 1\}$ and indeed for $0 < t < \min\{\tau^2 F(1), 1\}$ as well. Now, it is classical that $\sum_{i \leq t} Y_i$ can be approximated sufficiently close by standard Brownian motion and combining this with (2.42) immediatly gives (2.41).

The reason for avoiding standard Brownian motion in the formulation of Theorem 2.2.1 and Theorm 2.2.2 was to make the assertions more lucid. In fact, we will use standard Brownian motion for the proofs of the theorems.

Remark 2.2.2. Theorem 2.2.1 and Theorem 2.2.2 entail several consequences (see [1], [24], [28], [33]). Theorem 2.2.1, for instance, implies the following functional central limit theorem that contains the central limit theorems of Theorem 1.3.3 and Theorem 1.3.5 in Section 1.3.2.

Corollary 2.2.1. *We have, as $n \longrightarrow \infty$,*

$$\left\{\frac{1}{\sqrt{\tau^2 F(n)}} X(\tau^2 F(n) t),\ 0 \leq t \leq 1\right\} \longrightarrow \{W(t),\ 0 \leq t \leq 1\}.$$

Furthermore, we get from Theorem 2.2.1 a Strassen's type version of the law of the iterated logarithm, which generalizes the law of the iterated logarithm of Theorem 1.3.5 in Section 1.3.2. In order to state it, let K denote Strassen's set, that is the set of all real-valued, absolutely continuous functions $g(t)$ defined on the interval $[0,1]$ satisfying the following conditions

$$g(0) = 0, \qquad \int_0^1 (g'(t))^2 dt \leq 1.$$

Then, we have the following corollary

Corollary 2.2.2. *The sequence of functions*

$$\frac{1}{(2\tau^2 F(n) \log\log F(n))^{1/2}} X(\tau^2 F(n) t)$$

defined on $[0,1]$ is a.s. relatively compact in the topology of uniform convergence and has K as its set of limit points.

From Theorem 2.2.2 we can get even more. As an example, we state a functional law of the iterated logarithm for the maximum (for other consequences see [25]). Therefore, let J be the set of extended (∞ included in the range), non-negative, non-decreasing functions $h(t)$, $0 \leq t < \infty$ which are right continuous, except possibly at zero and satisfy

$$h(0) = 0, \qquad \int_0^\infty h^{-2}(t)dt \leq 1.$$

With this notation, we have

Corollary 2.2.3. *The sequence of functions*

$$\left(\frac{8 \log\log F(n)}{\pi^2\tau^2 F(n)}\right)^{1/2} \sup_{s\leq t} |X(\tau^2 F(n)s)|$$

defined on $[0,\infty)$ is a.s. relatively compact in the topology of weak convergence (i.e., pointwise convergence at all continuity points of the limit function) and has J as its set of limit points.

Remark 2.2.3. The central limit theorem contained in Corollary 2.2.1 can be seen as an approximation to the central limit theorem of Theorem 2.1.1 in Section 2.1.1.

This is getting even more lucid if we point out the following asymptotic expansions for σ and τ^2

Lemma 2.2.1. *We have*

$$\sigma = \frac{2}{r} + O\left(\frac{1}{d}\right),$$

and

$$\tau^2 = \frac{24(s,r)\varphi(r)\log d}{\pi^2 r C(s,r)} + O(1),$$

where

$$C(s,r) = r^2 \prod_{p|r}\left(1 - \frac{1}{p}\right) \prod_{p|\frac{r}{(s,r)}}\left(1 + \frac{1}{p}\right).$$

Especially, notice that the main term in the asymptotic expansion of σ is, according to Theorem 2.1.1, the constant belonging to the meanvalue of (2.3) and that τ^2 is increasing logarithmically with d, which could be seen as explanation for $F(n)\log F(n)$ to be the order of magnitude of the variance of (2.3).

The next three sections are dedicated to the proofs of Theorem 2.2.1 and Theorem 2.2.2 that will be performed in several steps (compare with the proof of Theorem 2.1.1 in Section 2.1.1). In a first step (Section 2.2.2), we are going to approximate the sequence $(X_n^{(d)})_{n\geq 1}$ several times. Then, we will prove the corresponding invariance principles for the approximating sequence (Section 2.2.3) and finally (Section 2.2.4), we are going to finish the proof by showing that the approximation is strong enough in order to entail Theorem 2.2.1 and Theorem 2.2.2 from the results for the approximating sequence proved in the section before.

2.2.2 Preliminaries

Consider the sequence of random variables

$$Y_k(x) := \#\{1 \leq c \leq d | cq_k \equiv s\ (r), c^2 \leq \varphi_k f(\log cq_k)\},$$

where $x \in [0,1]$. We need the following simple fact

Lemma 2.2.2. *We have*

$$\sum_{q_{k+1}\leq n} Y_k(x) + O(1) \leq X_n^{(d)}(x) \leq \sum_{q_k \leq n} Y_k(x) + O(1). \tag{2.43}$$

Proof. It is plain, by elementary properties of the continued fraction expansion, that we have

$$\left| x - \frac{p_k}{q_k} \right| \leq \frac{f(\log cq_k)}{cq_k^2} \quad \Longleftrightarrow \quad c^2 \leq \varphi_k f(\log cq_k).$$

Furthermore, if p/q is a solution of (1.23) with $(p,q) \leq d$ then it follows

$$\left| x - \frac{p}{q} \right| \leq \frac{f(\log q)}{q^2} \leq \frac{1}{2q^2}$$

for q large enough. Hence, by using another elementary property of the continued fraction expansion, there are integers $k \geq 0$ and $1 \leq c \leq d$ such that $p = cp_k$ and $q = cq_k$. Therefore, it is enough to count multiples cp_k/cq_k of the convergents of x with the restriction

$$1 \leq c \leq d,\ cq_k \equiv s\ (r),\ \text{and}\ c^2 \leq \varphi_k f(\log cq_k), \tag{2.44}$$

in the definition of X_n. This explains the right hand side of (2.43).

For the left hand side of (2.43) suppose that cp_k/cq_k satisfies (2.44) and $q_{k+1} \leq n$. By the elementary inequality

$$\sqrt{1/4(x+2)} \leq x, \quad \text{for } x \geq 1$$

we have

$$c \leq \sqrt{\varphi_k f(\log cq_k)} \leq \sqrt{1/4(a_{k+1}+2)} \leq a_{k+1}$$

for k large enough and therefore

$$cq_k \leq a_{k+1}q_k \leq q_{k+1} \leq n.$$

This proves the left hand side of (2.43). $\square$

Next, we apply Theorem 1.2.5 of Section 1.2.1. Using this Theorem, we have for each $\epsilon > 0$ that there exist κ large enough such that

$$k \log \gamma - \kappa k^{1-\delta_1} \leq \log q_k \leq k \log \gamma + \kappa k^{1-\delta_1}, \quad k \geq 1 \tag{2.45}$$

for a subset F of $[0,1]$ with $\lambda(F) \geq 1 - \epsilon$. Consequently, we get

$$\begin{aligned} f\big((k+1)\log \gamma + \kappa(k+1)^{1-\delta_1}\big) \leq f(\log q_{k+1}) &\leq f(\log c q_k) \\ &\leq f(\log q_k) \leq f(k \log \gamma - \kappa k^{1-\delta_1}) \end{aligned}$$

for $x \in F$ and $1 \leq c \leq a_{k+1}$.

We set

$$f_1(k) := f\big((k+1)\log \gamma + \kappa(k+1)^{1-\delta_1}\big), \quad f_2(k) := f(k \log \gamma - \kappa k^{1-\delta_1}),$$

and

$$F_i(n) = \sum_{k=1}^{n} f_i(k), \quad i = 1, 2.$$

Then, we define

$$\begin{aligned} \varphi_k^{(i)} = a_{k+1} &+ [0, a_{k+2}, \ldots, a_{k+[c_k^{(i)} \log F_i(k)]}] \\ &+ [0, a_k, a_{k-1}, \ldots, a_{k-[c_k^{(i)} \log F_i(k)]}], \quad i = 1, 2, \end{aligned}$$

where $c_k^{(i)}$ is chosen uniformly bounded and in such a fashion that $c_k^{(i)} \geq 8/\ln 2$, $[c_k^{(1)} \log F_1(k)]$ is odd, and $[c_k^{(2)} \log F_2(k)]$ is even. With this notation, we consider the following random variables

$$Z_k^{(i)}(x) := \#\{1 \leq c \leq d | c q_k \equiv s\ (r), c^2 \leq \varphi_k^{(i)} f_i(k)\}, \quad i = 1, 2.$$

By the definition of $\varphi_k^{(i)}$, we have

$$\varphi_k^{(1)} \leq \varphi_k \leq \varphi_k^{(2)}$$

and together with the definition of $f_i(k)$, we get

$$Z_k^{(1)}(x) \leq Y_k(x) \leq Z_k^{(2)}(x) \tag{2.46}$$

for $x \in F$ and k large enough.

Next, we need the following technical lemma that is straighforward to prove

Lemma 2.2.3. *If f satisfies (1.25), (1.26), and (2.39) then, we have*

$$\sum_{k=1}^{n} f_i^2(k) = O(F_i(n)^{1-\delta_3}). \tag{2.47}$$

We shall use the last lemma together with Theorem 1.2.8 of Section 1.2.2 and some ideas of [23] in order to obtain the following result

Lemma 2.2.4. *We have*

$$\mathbf{E}\sum_{k=1}^{n} Z_k^{(i)} = (\sigma \log\gamma) F_i(n) + O(1), \tag{2.48}$$

and either

$$\mathbf{V}\sum_{k=1}^{n} Z_k^{(i)} \sim (\tau^2 \log\gamma) F_i(n), \tag{2.49}$$

if f satisfies (1.25) and (1.26), or

$$\mathbf{V}\sum_{k=1}^{n} Z_k^{(i)} = (\tau^2 \log\gamma) F_i(n) + O(F_i(n)^{1-\delta_3}), \tag{2.50}$$

if f satisfies (1.25), (1.26), and (2.39).

Proof. We will only consider the case $i = 1$ because $i = 2$ can be treated in the same fashion.

Define a sequence of random variables as

$$U_k^{(1)}(x) := \#\{1 \le c \le d | cq_k \equiv s\ (r), c^2 \le \varphi_k f_1(k)\}.$$

Using Theorem 1.2.8 of Section 1.2.2, we get

$$\begin{aligned} \mathbf{E}U_k^{(1)} &= \sum_{0\le q<r} \sum_{\substack{1\le c\le d\\ cq\equiv s\ (r)}} \frac{1}{C(r)\ln 2} \frac{r\varphi((q,r))}{(q,r)} \frac{f_1(k)}{c^2}(1+\mathrm{O}(q^k)) \\ &= (\sigma\log\gamma) f_1(k)(1+\mathrm{O}(q^k)) \end{aligned} \tag{2.51}$$

and therefore

$$\mathbf{E}\sum_{k=1}^{n} U_k^{(1)} = (\sigma \log\gamma) F_1(n) + \mathrm{O}(1). \tag{2.52}$$

Furthermore, observe

$$\varphi_k - \varphi_k^{(1)} \ll 2^{-1/2c_k^{(1)} \log F_1(k)} \ll F_1(k)^{-4}$$

which yields

$$\begin{aligned} \lambda[U_k^{(1)} \ne Z_k^{(1)}] &\le \lambda[\exists c : 1 \le c \le d | \varphi_k^{(1)} f_1(k) < c^2 \le \varphi_k f_1(k)] \\ &\ll \sum_{c=1}^{d} \frac{f_1(k)}{c^2} - \left(\frac{c^2}{f_1(k)} + \mathrm{O}(F_1(k)^{-4})\right)^{-1} \\ &\ll \frac{f_1(k)^2}{F_1(k)^4} \end{aligned} \tag{2.53}$$

and hence

$$\mathbf{E}(U_k^{(1)} - Z_k^{(1)}) \ll \frac{f_1(k)^2}{F_1(k)^4}. \tag{2.54}$$

Thus, by taking the Dini-Abel Theorem into account, we have

$$\sum_{k=1}^{n} \mathbf{E}(U_k^{(1)} - Z_k^{(1)}) \ll 1 \tag{2.55}$$

and by combining (2.52) and (2.55), we get (2.48) as a consequence.

For the proof of (2.49) and (2.50) observe

$$\begin{aligned}\mathbf{E}(U_k^{(1)})^2 &= \sum_{0\le q<r} \sum_{\substack{1 \le c \le d \\ cq \equiv s\ (r)}} \frac{k_{c,q}}{C(r)\ln 2} \frac{r\varphi((q,r))}{(q,r)} \frac{f_1(k)}{c^2}(1+\mathrm{O}(q^k)) \\ &= (\tau^2 \log\gamma) f_1(k)(1+\mathrm{O}(q^k)).\end{aligned} \tag{2.56}$$

As in [23] (proof of Lemma 3.1.1) it is quite easy to see that we have

$$\mathrm{Cov}(U_{k_1}^{(1)} U_{k_2}^{(1)}) \ll f_1^2(k_1)\rho^{k_2-k_1}, \tag{2.57}$$

for $k_1 < k_2$ where $\rho < 1$ is a real constant. By combining (2.51), (2.56), (2.57), and using Lemma 2.2.3, we get

$$\mathbf{V}\sum_{k=1}^{n} U_k^{(1)} = \sum_{k=1}^{n} \mathbf{E}(U_k^{(1)})^2 - \sum_{k=1}^{n}(\mathbf{E}U_k^{(1)})^2 + 2 \sum_{1\le k_1<k_2\le n} \mathrm{Cov}(U_{k_1}^{(1)} U_{k_2}^{(1)})$$

which is either asymptotic

$$(\tau^2 \log\gamma) F_1(n)$$

or equal to

$$(\tau^2 \log\gamma) F_1(n) + \mathrm{O}(F_1(n)^{1-\delta_3})$$

depending on the assumptions on f.

Finally, notice

$$\mathbf{E}(U_k^{(1)} - Z_k^{(1)})^2 \ll \frac{f_1(k)^2}{F_1(k)^4} \tag{2.58}$$

which is plain by (2.53) and therefore, we can proceed as in [23] in order to obtain (2.49) and (2.50). □

The next section is devoted to the proofs of invariance principles corresponding to (2.37), (2.38), and (2.40) for $Z_k^{(i)}$.

2.2.3 Invariance Principles for $Z_k^{(i)}$

Throughout this section, we will suppress the dependence on i. Furthermore, we use

$$\xi_k = Z_k - \mathbf{E}Z_k \tag{2.59}$$

for the centered version of Z_k.

In the following, we want to apply Theorem 1.1.2 of Section 1.1.2 to the sequence $(\xi_k)_{k\geq 1}$. Because of the assumptions (1.6) resp. (1.9) it is not possible to apply the theorem directly (compare with (2.49) resp. (2.50)). So, before we use Theorem 1.1.2, we need a suitable blocking.

Therefore, we define for a positive integer k the integer h_k by

$$(\tau^2 \log\gamma)F(h_k) \leq k < (\tau^2 \log\gamma)F(h_k + 1).$$

Furthermore, we put $h_0 = 0$ and consider a sequence of random variables defined as

$$\eta_k = \sum_{l=h_{k-1}+1}^{h_k} \xi_l$$

for $k \geq 1$. The sequence $(\eta_k)_{k\geq 1}$ satisfies the assumptions of Theorem 1.1.2 according to the following lemma

Lemma 2.2.5. *We have*

1. *The sequence $(\eta_k)_{k\geq 1}$ satisfies a retarded strong mixing condition of the form (1.3) with arbitrary $\kappa > 0$ and $\varphi(s) = q^s, q < 1$.*

2. *Either*

$$\mathbf{E}\left(\sum_{k=1}^{n} \eta_k\right)^2 \sim n,$$

   if f satisfies (1.25) and (1.26), or

$$\mathbf{E}\left(\sum_{k=1}^{n} \eta_k\right)^2 = n + O(n^{1-\delta_3}),$$

   if f satisfies (1.25), (1.26), and (2.39).

3. $$\mathbf{E}|\eta_k|^3 \ll \log k.$$

Proof. We start by pointing out that the sequence $(\xi_k)_{k\geq 1}$ satisfies a mixing condition of the form

$$|\lambda(A\cap B) - \lambda(A)\lambda(B)| \ll q^{k-\log F(t)} \tag{2.60}$$

for all $A \in \mathcal{F}_1^t$ and all $B \in \mathcal{F}_{t+k}^\infty$, where $q < 1$ is the constant in Theorem 1.2.8 in Section 1.2.2. This follows immediatly from a result of Szüsz [30] (compare also with 3.3.4 in [23]). Let $\mathcal{M}_a^b$ be the σ-algebra generated by $\eta_k, a \leq k \leq b$. Then, we have for $A \in \mathcal{M}_1^t$ and $B \in \mathcal{M}_{t+k}^\infty$

$$|\lambda(A \cap B) - \lambda(A)\lambda(B)| \ll q^{h_{t+k} - h_t - \log F(h_t)}. \tag{2.61}$$

Furthermore, observe

$$0 \leq t - (\tau^2 \log \gamma)F(h_t) < (\tau^2 \log \gamma)f(h_t + 1) \ll 1 \tag{2.62}$$

and together with (2.61), we get

$$|\lambda(A \cap B) - \lambda(A)\lambda(B)| \ll q^{k - \log t}.$$

It is plain that this implies

$$|\lambda(A \cap B) - \lambda(A)\lambda(B)| \ll q^{kt^{-\kappa}}$$

for arbitrary $\kappa > 0$ and therefore (1) is proved.

(2) follows easily by using (2.49) resp. (2.50) and (2.62)

$$\mathbf{E}\left(\sum_{k=1}^{n} \eta_k\right)^2 = \mathbf{E}\left(\sum_{k=1}^{h_n} \xi_k\right)^2$$

which is either asymptotic

$$(\tau^2 \log \gamma)F(h_n) \sim n,$$

or equal to

$$(\tau^2 \log \gamma)F(h_n) + \mathrm{O}(F(h_n)^{1-\delta_3}) = n + \mathrm{O}(n^{1-\delta_3})$$

depending on the assumptions on f.

In order to prove (3), we observe that for $l \geq 1$

$$\mathbf{E}|\xi_k|^l \ll \mathbf{E}Z_k \ll f(k) \tag{2.63}$$

where (2.51) and (2.54) was used. Then, we expand the left hand side of (3) by the multinomial theorem

$$\begin{aligned} \mathbf{E}|\eta_k|^3 &= \mathbf{E}\left|\sum_{l=h_{k-1}+1}^{h_k} \xi_l\right|^3 \\ &= \sum_{e_{h_{k-1}+1}+\ldots+e_{h_k}=3} \binom{3}{e_{h_{k-1}+1}, \ldots, e_{h_k}} \mathbf{E}|\xi_{h_{k-1}+1}|^{e_{h_{k-1}+1}} \cdots |\xi_{h_k}|^{e_{h_k}} \end{aligned}$$

and break the last sum into several parts according to the powers in the product $|\xi_{h_{k-1}+1}|^{e_{h_{k-1}+1}}\cdots|\xi_{h_k}|^{e_{h_k}}$.

For the third powers, we get by (2.62) and (2.63)

$$\sum_{l=h_{k-1}+1}^{h_k}\mathbf{E}|\xi_l|^3 \ll \sum_{l=h_{k-1}+1}^{h_k} f(l) = F(h_k)-F(h_{k-1}) \ll 1.$$

Next, we consider

$$\sum_{h_{k-1}<l_1<l_2\le h_k}\mathbf{E}|\xi_{l_1}|^2|\xi_{l_2}| \ll \sum_{h_{k-1}<l_1<l_2\le h_k}\mathbf{E}|\xi_{l_1}||\xi_{l_2}|. \tag{2.64}$$

Define random variables as

$$\zeta_k = U_k - \mathbf{E}U_k$$

where U_k is defined in the proof of Lemma 2.2.4 and

$$\zeta_{k,\lambda} = U_{k,\lambda} - \mathbf{E}U_{k,\lambda}$$

where $U_{k,\lambda}$ is defined as follows

$$U_{k,\lambda}(x) := \#\{1\le c\le d|cq_k\equiv s\ (r), c^2\le \varphi_k^{(\lambda)} f(k)\}$$

with

$$\varphi_k^{(\lambda)} = a_{k+1}+[a_{k+2},\dots,a_{k+\lambda}]+[a_k,\dots,a_{k-\lambda}].$$

Similarly to (2.53), it is plain that

$$\lambda[U_k\neq U_{k,\lambda}] \ll f(k)^2 2^{-\lambda/2}$$

and hence

$$\mathbf{E}|\zeta_k-\zeta_{k,\lambda}| \ll f(k)^2 2^{-\lambda/2}. \tag{2.65}$$

Furthermore, if we chose for integers $k_1<k_2$

$$\lambda = \left[\frac{k_2-k_1}{3}\right]$$

then, we get

$$\mathbf{E}|\zeta_{k_1,\lambda}||\zeta_{k_2,\lambda}| \ll f(k_1)f(k_2). \tag{2.66}$$

This follows by Lemma 1.1.1 in Section 1.2.1 together with a result due to Szüsz (see [30]) about the mixing behaviour of the sequence $\zeta_{k,\lambda}$. If we combine (2.65) and (2.66), we obtain

$$\begin{aligned}\mathbf{E}|\zeta_{k_1}||\zeta_{k_2}| &\ll \mathbf{E}|\zeta_{k_1}-\zeta_{k_1,\lambda}|+\mathbf{E}|\zeta_{k_2}-\zeta_{k_2,\lambda}|+\mathbf{E}|\zeta_{k_1,\lambda}||\zeta_{k_2,\lambda}|\\ &\ll f(k_1)\rho^{k_2-k_1}+f(k_2)\rho^{k_2-k_1}+f(k_1)f(k_2),\end{aligned}$$

with a constant $\rho < 1$, where λ is defined as above. Thus, by (2.62) and the last estimate,

$$\sum_{h_{k-1}<l_1<l_2\le h_k} \mathbf{E}|\zeta_{l_1}||\zeta_{l_2}| \ll \sum_{l=h_{k-1}+1}^{h_k} f(l) + \left(\sum_{l=h_{k-1}+1}^{h_k} f(l)\right)^2 \ll 1$$

and therefore

$$\begin{aligned} \mathbf{E}\left(\sum_{l=h_{k-1}+1}^{h_k} |\zeta_l|\right)^2 &= \sum_{l=h_{k-1}+1}^{h_k} \mathbf{E}|\zeta_l|^2 + 2 \sum_{h_{k-1}<l_1<l_2\le h_k} \mathbf{E}|\zeta_{l_1}||\zeta_{l_2}| \\ &\ll \sum_{l=h_{k-1}+1}^{h_k} f(l) + 1 \ll 1, \end{aligned} \tag{2.67}$$

where (2.51) was used. Next, observe

$$\begin{aligned} &\mathbf{E}\left(\sum_{l=h_{k-1}+1}^{h_k} |\xi_l|\right)^2 - \mathbf{E}\left(\sum_{l=h_{k-1}+1}^{h_k} |\zeta_l|\right)^2 \\ &\ll \mathbf{E}^{1/2}\Big(\sum_{l=h_{k-1}+1}^{h_k} (|\xi_l| - |\zeta_l|)\Big)^2 \cdot \\ &\quad \cdot \left(\mathbf{E}^{1/2}\Big(\sum_{l=h_{k-1}+1}^{h_k} (|\xi_l| - |\zeta_l|)\Big)^2 + \mathbf{E}^{1/2}\Big(\sum_{l=h_{k-1}+1}^{h_k} |\zeta_l|\Big)^2\right) \end{aligned} \tag{2.68}$$

and by using (2.58)

$$\begin{aligned} \mathbf{E}\Big(\sum_{l=h_{k-1}+1}^{h_k} (|\xi_l| - |\zeta_l|)\Big)^2 &\le \sum_{h_{k-1}<l_1,l_2\le h_k} \mathbf{E}|\xi_{l_1} - \zeta_{l_1}||\xi_{l_2} - \zeta_{l_2}| \\ &\le \sum_{h_{k-1}<l_1,l_2\le h_k} \mathbf{E}^{1/2}(\xi_{l_1} - \zeta_{l_1})^2 \mathbf{E}^{1/2}(\xi_{l_2} - \zeta_{l_2})^2 \\ &\ll \sum_{h_{k-1}<l_1,l_2\le h_k} \frac{f(l_1)}{F(l_1)^2}\frac{f(l_2)}{F(l_2)^2} \ll 1, \end{aligned} \tag{2.69}$$

where the last estimate follows by the Dini-Abel Theorem.

Finally by combining (2.67), (2.68), and (2.69), we get for the right hand side of (2.64)

$$\sum_{h_{k-1}<l_1<l_2\le h_k} \mathbf{E}|\xi_{l_1}||\xi_{l_2}| \le \mathbf{E}\left(\sum_{l=h_{k-1}+1}^{h_k} |\xi_l|\right)^2 \ll 1. \tag{2.70}$$

The next sum
$$\sum_{h_{k-1}<l_1<l_2\le h_k} \mathbf{E}|\xi_{l_1}||\xi_{l_2}|^2$$
is treated in the same manner.

In order to prove (3) we are left with the following sum
$$\sum_{h_{k-1}<l_1<l_2<l_3\le h_k} \mathbf{E}|\xi_{l_1}||\xi_{l_2}||\xi_{l_3}|.$$

We break the sum into two parts $\sum = \sum^* + \sum^{**}$ according to whether $l_3 - l_2 > [c_{l_2}\log F(l_2)]$ or not. In the first case the random variables $|\xi_{l_1}||\xi_{l_2}|$ and $|\xi_{l_3}|$ are satisfying a mixing condition which is stronger than (2.60), namely
$$|\lambda(AB) - \lambda(A)\lambda(B)| \ll q^{l_3-l_2-\log F(l_2)}\lambda(A)\lambda(B)$$
for all A in the σ-algebra generated by $|\xi_{l_1}||\xi_{l_2}|$ and all B in the σ-algebra generated by $|\xi_{l_3}|$. This is a consequence of a theorem due to Szüsz in [30]. Therefore, we can use Lemma 1.1.1 in 1.2.1 and obtain
$$\begin{aligned}
\sum_{h_{k-1}<l_1<l_2<l_3\le h_k}^{*} \mathbf{E}|\xi_{l_1}||\xi_{l_2}||\xi_{l_3}| &\ll \sum_{h_{k-1}<l_1<l_2<l_3\le h_k}^{*} \mathbf{E}|\xi_{l_1}||\xi_{l_2}|\mathbf{E}|\xi_{l_3}| \\
&\ll \left(\sum_{h_{k-1}<l_1<l_2\le h_k} \mathbf{E}|\xi_{l_1}||\xi_{l_2}|\right)\left(\sum_{l=h_{k-1}+1}^{h_k} \mathbf{E}|\xi_l|\right) \\
&\ll \sum_{l=h_{k-1}+1}^{h_k} f(l) \ll 1,
\end{aligned}$$
where (2.62), (2.63), and (2.70) was used. For the last sum we get, by another application of (2.62) and (2.70),
$$\begin{aligned}
\sum_{h_{k-1}<l_1<l_2<l_3\le h_k}^{**} \mathbf{E}|\xi_{l_1}||\xi_{l_2}||\xi_{l_3}| &\ll \sum_{h_{k-1}<l_1<l_2\le h_k} \log F(l_2)\mathbf{E}|\xi_{l_1}||\xi_{l_2}| \\
&\ll \log F(h_k) \ll \log k,
\end{aligned}$$
which conludes the proof of (3). $\square$

Because of Lemma 2.2.5, we can apply Theorem 1.1.2 of Section 1.2.1 to the sequence $(\eta_k)_{k\ge1}$ in order to obtain, as $t \longrightarrow \infty$, either
$$Y(t) - W(t) = \mathrm{o}((t\log\log t)^{1/2}), \text{ a.s.} \tag{2.71}$$
and
$$\lambda_2\left[\frac{1}{\sqrt{t}}\sup_{s\le t}|Y(s) - W(s)| \ge \epsilon\right] \longrightarrow 0 \tag{2.72}$$

for all $\epsilon > 0$, or

$$Y(t) - W(t) \ll t^{1/2-\delta} \text{ a.s.} \tag{2.73}$$

for a suitable constant δ, depending on the assumptions on f. Here, the stochastic process $\{Y(t), t \geq 0\}$ is defined on the probability space $([0,1]^2, \mathcal{B}_2, \lambda_2)$ as

$$Y(t) = \sum_{k \leq t} \eta_k.$$

Our next aim is to show that (2.71), (2.72), and (2.73) for η_k entail the corresponding results for ξ_k. Therefore, consider for a real number $t \geq 0$

$$n_t = \begin{cases} \max\{n | (\tau^2 \log \gamma) F(n) \leq t\} & \text{if } t \geq (\tau^2 \log \gamma) F(1), \\ 0 & \text{otherwise,} \end{cases}$$

and define a stochastic process $\{Z(t), t \geq 0\}$ on the probability space $([0,1]^2, \mathcal{B}_2, \lambda_2)$ as

$$Z(t) = \sum_{k \leq n_t} \xi_k. \tag{2.74}$$

Lemma 2.2.6. *We have, as $t \longrightarrow \infty$,*

$$Z(t) - Y(t) \ll t^{1/2-\epsilon} \text{ a.s.}, \tag{2.75}$$

for all $\epsilon < 1/6$.

Proof. Since

$$\begin{aligned} \lambda_2 \left[\sum_{l=h_{k-1}+1}^{h_k} |\xi_l| \geq k^{1/2-\epsilon} \right] &= \lambda \left[\sum_{l=h_{k-1}+1}^{h_k} |\xi_l| \geq k^{1/2-\epsilon} \right] \\ &\leq \mathbf{E} \left(\sum_{l=h_{k-1}+1}^{h_k} |\xi_l| \right)^3 / k^{3/2-3\epsilon} \ll \frac{\log k}{k^{3/2-3\epsilon}} \end{aligned}$$

because of the proof of Lemma 2.2.5 (3), we get from the convergence part of the Borel-Cantelli Lemma

$$\sum_{l=h_{k-1}+1}^{h_k} |\xi_l| \ll k^{1/2-\epsilon} \text{ a.s..} \tag{2.76}$$

Next, observe

$$|Z(t) - Y(t)| = \left| \sum_{k \leq n_t} \xi_k - \sum_{k \leq h_{[t]}} \xi_k \right| \leq \sum_{l=h_{[t]}+1}^{h_{[t]+1}} |\xi_l|$$

which together with (2.76) proves the Lemma. $\square$

By combining (2.71), (2.72), (2.73), and (2.75), we finally get

Lemma 2.2.7. *We have, as $t \longrightarrow \infty$, either*

$$Z(t) - W(t) = o((t \log\log t)^{1/2}) \ a.s. \tag{2.77}$$

and

$$\lambda_2 \left[\frac{1}{\sqrt{t}} \sup_{s \le t} |Z(s) - W(s)| \ge \epsilon \right] \longrightarrow 0 \tag{2.78}$$

for all $\epsilon > 0$, or

$$Z(t) - W(t) \ll t^{1/2-\lambda} \ a.s. \tag{2.79}$$

for all $\lambda \le \min\{\delta, 1/6\}$, depending on the assumptions on f.

Proof. The proof of (2.77) and (2.79) is obvious. In order to prove (2.78) observe that because of (2.75) there exist for each $\rho < 1$ a subset E of $[0,1]^2$ with measure at least $1-\rho$ and κ large enough such that for all elements in E and for all $t \ge 0$, we have

$$Z(t) - Y(t) \le \kappa t^{1/2-\bar{\epsilon}}$$

where $\bar{\epsilon} < 1/6$. Hence

$$\begin{aligned} \frac{1}{\sqrt{t}} \sup_{s \le t} |Z(s) - W(s)| &\le \frac{1}{\sqrt{t}} \sup_{s \le t} (|Z(s) - Y(s)| + |Y(s) - W(s)|) \\ &\le \frac{1}{\sqrt{t}} \sup_{s \le t} |Y(s) - W(s)| + \frac{\kappa}{t^{\bar{\epsilon}}} \end{aligned}$$

for all elements in E. Therefore, we have

$$\left[\frac{1}{\sqrt{t}} \sup_{s \le t} |Z(s) - W(s)| \ge \epsilon \right] \cap E \subseteq \left[\frac{1}{\sqrt{t}} \sup_{s \le t} |Y(s) - W(s)| \ge \epsilon/2 \right]$$

for t large enough. By taking (2.72) into account, we get, as $t \longrightarrow \infty$,

$$\lambda_2 \left(\left[\frac{1}{\sqrt{t}} \sup_{s \le t} |Z(s) - W(s)| \ge \epsilon \right] \cap E \right) \longrightarrow 0. \tag{2.80}$$

Thus, by the trivial inequality $\lambda_2(A \cap B) \ge \lambda_2(A) + \lambda_2(B) - 1$,

$$\lambda_2 \left(\left[\frac{1}{\sqrt{t}} \sup_{s \le t} |Z(s) - W(s)| \ge \epsilon \right] \cap E \right) \ge \lambda_2 \left[\frac{1}{\sqrt{t}} \sup_{s \le t} |Z(s) - W(s)| \ge \epsilon \right] - \rho$$

which together with (2.80) gives

$$\lambda_2 \left[\frac{1}{\sqrt{t}} \sup_{s \le t} |Z(s) - W(s)| \ge \epsilon \right] \le 2\rho$$

for t large enough and hence (2.78) is proved. □

In the next section, we will finish the proofs of our Theorem 2.2.1 and Theorem 2.2.2 by showing that the sequence Z_k approximates X_n good enough in order to obtain the corresponding invariance principles for X_n from Lemma 2.2.7.

2.2.4 Proof of Theorem 2.2.1 and Theorem 2.2.2

Before we can start proving the theorems, we need few additional observations. First the following Lemma about standard Brownian motion

Lemma 2.2.8. *Let $\{W(t), t \geq 0\}$ be standard brownian motion on some probability space $(\Omega, \mathcal{A}, P)$ and $0 < \delta < 1$. Then, we have, as $t \longrightarrow \infty$,*

$$W(t + O(t^{1-\delta})) - W(t) \ll t^{(1-\delta)/2+\lambda} \quad a.s.$$

for all $\lambda > 0$.

Proof. Consider

$$\sup_{n \leq t < n+1} |W(t + \mathrm{O}(t^{1-\delta})) - W(t)| \leq \sup_{a \leq t, s \leq b} |W(t) - W(s)| := R(a, b),$$

where $a = n - Cn^{1-\delta}, b = (n+1) + C(n+1)^{1-\delta}$ and C is a suitable constant.
The following estimation is elementary

$$P[R(a, b) \geq n^{(1-\delta)/2+\lambda}] = P[R(0, 1) \geq C_1 n^{\lambda}] \ll e^{-C_2 n^{2\lambda}}$$

where C_1, C_2 are positive constants. Hence, applying the Borel-Cantelli Lemma immediatly gives the result. □

Furthermore, we need a technical lemma due to Philipp (see [23] and compare with Lemma 2.1.7 in Section 2.1.4).

Lemma 2.2.9. *Let g_1 (resp. g_2) be the inverse function of $\gamma^{(k+1)} \exp(\kappa(k+1)^{1-\delta_1})$ (resp. $\gamma^k exp(-\kappa k^{1-\delta_1})$). Then, we have*

$$F_i(g_i(n)) = \frac{1}{\log \gamma} F(n) + O(F(n)^{1/2-\delta_2}).$$

Now, we are ready for the proof of Theorem 2.2.1 and Theorem 2.2.2.

Proof of Theorem 2.2.1 and Theorem 2.2.2 . First observe, by (2.43), (2.46), and Lemma 2.2.9,

$$X_n^{(d)}(x) \leq \sum_{q_k \leq n} Y_k(x) + \mathrm{O}(1) \leq \sum_{k \leq g_2(n)} Y_k(x) + \mathrm{O}(1) \leq \sum_{k \leq g_2(n)} Z_k^{(2)}(x) + \mathrm{O}(1)$$

and

$$\sum_{k \leq g_1(n)} Z_k^{(1)}(x) + \mathrm{O}(1) \leq \sum_{k \leq g_1(n)} Y_k(x) + \mathrm{O}(1) \leq \sum_{q_{k+1} \leq n} Y_k(x) + \mathrm{O}(1) \leq X_n^{(d)}(x)$$

for all $x \in F$.

Thus, for all $x_1 \in F$ and $x_2 \in [0,1]$,

$$\begin{aligned} X(t) - W(t) &= X_{n_t}^{(d)} - \sigma F(n_t) - W(t) \\ &\le \sum_{k \le g_2(n_t)} Z_k^{(2)} - \sigma F(n_t) - W(t) + \mathrm{O}(1) \\ &= \sum_{k \le g_2(n_t)} \xi_k^{(2)} - W(t) + \mathrm{O}(F(n_t)^{1/2-\delta_2}) + \mathrm{O}(1) \qquad (2.81) \\ &= Z^{(2)}(\tau^2 F(n_t) + \mathrm{O}(F(n_t)^{1/2-\delta_2})) \\ &\qquad\qquad - W(t) + \mathrm{O}(F(n_t)^{1/2-\delta_2}) + \mathrm{O}(1) \end{aligned}$$

where Lemma 2.2.4 and Lemma 2.2.9 was used and $\xi_k^{(2)}$ resp. $Z^{(2)}(t)$ is defined by (2.59) resp. (2.74). Next consider

$$0 \le t - \tau^2 F(n_t) \le \tau^2 \frac{f(\log n_t + 1)}{n_t + 1} \ll 1$$

and therefore

$$\tau^2 F(n_t) = t + \mathrm{O}(1). \qquad (2.82)$$

Furthermore, we have by Lemma 2.2.8

$$W(t + \mathrm{O}(t^{1/2-\delta_2})) - W(t) \ll t^{1/2-\delta} \text{ a.s.} \qquad (2.83)$$

for a suitable constant $0 < \delta < 1/2$.

If we combine (2.81), (2.82), and (2.83) then, we obtain that for each $\epsilon > 0$ there exists a subset E of $[0,1]^2$ with measure at least $1 - \epsilon$ so that for κ large enough (and only depending on E), we have

$$\begin{aligned} X(t) - W(t) \le Z^{(2)}(t + \mathrm{O}(t^{1/2-\delta_2})) &- W(t + \mathrm{O}(t^{1/2-\delta_2})) \\ &+ \mathrm{O}(t^{1/2-\lambda}) + \mathrm{O}(1) \end{aligned} \qquad (2.84)$$

for all elements in E and $t \ge 0$, where λ is a suitable constant only depending on δ_2 (thereby notice that the definition of $Z^{(2)}$ is depending on κ). Similarly, we get

$$\begin{aligned} X(t) - W(t) \ge Z^{(1)}(t + \mathrm{O}(t^{1/2-\delta_2})) &- W(t + \mathrm{O}(t^{1/2-\delta_2})) \\ &+ \mathrm{O}(t^{1/2-\lambda}) + \mathrm{O}(1) \end{aligned} \qquad (2.85)$$

for all elements in E.

The proof of (2.37) is now an easy consequence of (2.77), (2.84), and (2.85).

In order to prove (2.38), we observe by combining (2.84) and (2.85)

$$\begin{aligned} |X(t) - W(t)| \le \max\{&|Z^{(2)}(t + \mathrm{O}(t^{1/2-\delta_2})) - W(t + \mathrm{O}(t^{1/2-\delta_2}))|, \\ &|Z^{(1)}(t + \mathrm{O}(t^{1/2-\delta_2})) - W(t + \mathrm{O}(t^{1/2-\delta_2}))|\} + \mathrm{O}(t^{1/2-\lambda}) + \mathrm{O}(1) \end{aligned}$$

for all elements in E. Hence, it follows that for each $\delta < \epsilon$ there exist a subset G of E with measure at least $1-\delta$ and a constant $\bar{\kappa}$ such that

$$\begin{aligned}|X(t)-W(t)| \le \max\{&|Z^{(2)}(t+\mathrm{O}(t^{1/2-\delta_2}))-W(t+\mathrm{O}(t^{1/2-\delta_2}))|,\\ &|Z^{(1)}(t+\mathrm{O}(t^{1/2-\delta_2}))-W(t+\mathrm{O}(t^{1/2-\delta_2}))|\}+\bar{\kappa}t^{1/2-\lambda}+\bar{\kappa}\end{aligned}$$

for all elements in G and $t \ge 0$. Thus,

$$\begin{aligned}\frac{1}{\sqrt{t}}\sup_{s\le t}|X(s)-W(s)| \le{}& \frac{1}{\sqrt{t}}\sup_{s\le t}|Z^{(2)}(s+\mathrm{O}(s^{1/2-\delta_2}))-W(s+\mathrm{O}(s^{1/2-\delta_2}))|\\ &+\frac{1}{\sqrt{t}}\sup_{s\le t}|Z^{(1)}(s+\mathrm{O}(s^{1/2-\delta_2}))-W(s+\mathrm{O}(s^{1/2-\delta_2}))|+\frac{\bar{\kappa}}{t^{\lambda}}+\frac{\bar{\kappa}}{\sqrt{t}}\end{aligned}$$

for all elements in G. Therefore, we get

$$\begin{aligned}&\left[\frac{1}{\sqrt{t}}\sup_{s\le t}|X(s)-W(s)| \ge \bar{\epsilon}\right]\cap G\\ &\quad\subseteq \left[\frac{1}{\sqrt{t}}\sup_{s\le t}|Z^{(2)}(s+\mathrm{O}(s^{1/2-\delta_2}))-W(s+\mathrm{O}(s^{1/2-\delta_2}))| \ge \bar{\epsilon}/8\right]\cup\\ &\qquad\cup\left[\frac{1}{\sqrt{t}}\sup_{s\le t}|Z^{(1)}(s+\mathrm{O}(s^{1/2-\delta_2}))-W(s+\mathrm{O}(s^{1/2-\delta_2}))| \ge \bar{\epsilon}/8\right]\end{aligned}$$

for t large enough. It is easy to see by (2.78) that

$$\lambda_2\left[\frac{1}{\sqrt{t}}\sup_{s\le t}|Z^{(i)}(s+\mathrm{O}(s^{1/2-\delta_2}))-W(s+\mathrm{O}(s^{1/2-\delta_2}))| \ge \bar{\epsilon}/8\right]\longrightarrow 0,$$

as $t\longrightarrow\infty$, for $i=1,2$. Hence

$$\lambda_2\left(\left[\frac{1}{\sqrt{t}}\sup_{s\le t}|X(s)-W(s)| \ge \bar{\epsilon}\right]\cap G\right)\le 2\delta$$

for t large enough and using the trivial inequality $\lambda(A\cap B)\ge\lambda(A)+\lambda(B)-1$ gives

$$\lambda_2\left[\frac{1}{\sqrt{t}}\sup_{s\le t}|X(s)-W(s)| \ge \bar{\epsilon}\right]\le 3\delta$$

for t large enough. This concludes the proof of (2.38).

For the proof of (2.40) observe by (2.79) that for $t\ge\tau^2F(1)$ the right hand side of (2.84) is bounded by $\bar{C}t^{1/2-\bar{\lambda}}$ for suitable $\bar{C}$ and $\bar{\lambda}$ and the same is true for $t<\tau^2F(1)$, because the brownian motion is locally Hölder contin- ues. A similar lower bound is obtained from (2.85) and therefore, Theorem 2.2.2 is proved. □

2.2.5 Proof of the Corollaries

In order to prove the Corollaries, we use the equivalent formulation of (2.37), (2.38), and (2.40) with standard Brownian motion $\{W(t), t \geq 0\}$.

Proof of Corollary 2.2.1. Observe that (2.38) implies

$$\frac{1}{\sqrt{\tau^2 F(n)}} \|X(\tau^2 F(n)t) - W(\tau^2 F(n)t)\| \xrightarrow{\lambda_2} 0$$

where $\|\cdot\|$ denotes the maximum norm. By pointing out

$$\{1/\sqrt{\tau^2 F(n)} W(\tau^2 F(n)t), 0 \leq t \leq 1\} \stackrel{\mathcal{L}}{=} \{W(t), 0 \leq t \leq 1\}$$

the result follows. □

Proof of Corollary 2.2.2. It is immediate from (2.37) that, as $n \longrightarrow \infty$,

$$\frac{1}{(2\tau^2 F(n) \log\log F(n))^{1/2}} \|X(\tau^2 F(n)t) - W(\tau^2 F(n)t)\| \longrightarrow 0 \text{ a.s.}$$

where $\|\cdot\|$ denotes the maximum norm.

Therefore, it is enough to prove the law of the iterated logarithm for the sequence of functions

$$\frac{1}{(2\tau^2 F(n) \log\log F(n))^{1/2}} W(\tau^2 F(n)t)$$

defined on $[0, 1]$. But because of

$$\tau^2 F(n+1) - \tau^2 F(n) \ll 1 \tag{2.86}$$

this follows by Strassen's method (see [28]). □

Proof of Corollary 2.2.3. (2.40) has the following simple consequence

$$\left(\frac{8 \log\log F(n)}{\pi^2 \tau^2 F(n)}\right)^{1/2} (\sup_{s \leq t} |X(\tau^2 F(n)s)| - \sup_{s \leq t} |W(\tau^2 F(n)s|) \longrightarrow 0 \text{ a.s.},$$

as $n \longrightarrow \infty$, pointwise in t. Therefore, it is enough to prove Corollary 2.2.3 for the functions

$$\left(\frac{8 \log\log F(n)}{\pi^2 \tau^2 F(n)}\right)^{1/2} \sup_{s \leq t} |W(\tau^2 F(n)s|$$

defined on the interval $[0, \infty)$. By taking (2.86) into account, for these functions the result follows from (4.6) in [33]. □

Proof of Lemma 2.2.1. For the first part, we use Lemma 2.1.1 in Section 2.1.2 in order to get

$$\sigma = \frac{2}{r} - \frac{12r}{C(r)\pi^2} \sum_{0 \le q < r} \frac{\varphi((q,r))}{(q,r)} \sum_{\substack{d < c \\ cq \equiv s\ (r)}} \frac{1}{c^2} = \frac{2}{r} + \mathrm{O}\left(\frac{1}{d}\right).$$

For the second part observe

$$k_{c,q} = \begin{cases} 2\frac{(q,r)}{r}c + c_q & \text{if } (q,r)|s, \\ 0 & \text{otherwise,} \end{cases}$$

with a suitable constand c_q. Hence

$$\begin{aligned} \tau^2 &= \frac{24}{C(r)\pi^2} \sum_{0 \le q < r, (q,r)|s} \varphi((q,r)) \sum_{\substack{1 \le c \le d \\ cq \equiv s\ (r)}} \frac{1}{c} + \mathrm{O}(1) \\ &= \frac{24 \log d}{C(r) r \pi^2} \sum_{0 \le q < r, (q,r)|s} \varphi((q,r))(q,r) + \mathrm{O}(1), \end{aligned}$$

which - together with Lemma 2.1.3 in Section 2.1.2 - proves the Lemma. □

Chapter 3

Metric Diophantine Approximation in the Formal Laurent Series Case

In this chapter, we are going to develop the metric theory of diophantine approximation in the field of formal Laurent series. Furthermore, we prove some results from the metric theory of continued fraction expansion in the Laurent series case, too (for instance an analogue of Theorem 1.2.6 of Section 1.2.2). Especially, we will see that most of the results stated in Chapter 1 and Chapter 2 posses analogues in the field of formal Laurent series.

Unlike the real case, in the formal Laurent series case, it is a big difference whether all solutions of inequality (1.23) (resp. the analogue inequality in the Laurent series case) are considered or if only solutions are taken into account that have denominators contained in an arithmetic progression. This is due to the fact that in the formal Laurent series case, the partial quotients form an independent and identically distributed sequence of random variables (compare with Corollary 1.4.2 and Corollary 1.4.3 in Section 1.4.2); in difference to the real case. Although, counting solutions that have denominator contained in an arithmetic progression is more general, we will start with the case of all solutions in order to point out the differences that occur in the Laurent series case.

3.1 Counting All Solutions - The Independent Case

One of the few papers that deal with the metric theory of diophantine approximation in the Laurent series case is the thesis of deMathan [2]. There, deMathan proved that Khintchine's Theorem of Section 1.3.1 possesses an analogue in the formal Laurent series case. Actually, deMathan already proved Khintchine's Theorem in higher dimensions. Here, we are only interested in the one dimensional case and it is well known that in this situation an easier proof can be derived from continued fraction theory.

In Section 3.1.2, we show that these arguments work in the formal Laurent series case, too. Then in Section 3.1.3, we give an answer to LeVeque's second problem in the formal Laurent series case (without the restriction that the denominators of the solutions have to be in an arithmetic progression); thereby, we will see that the proof is much easier than in the real case. Finally in Section 3.1.4, we will provide an analogue of Theorem 1.3.3 of Section 1.3.2. Some of the results are either taken from [9] or constitute generalizations of results in [9]. The next section will be used in order to state the results.

3.1.1 Statement of the Results

In the following, we are interested in statistical properties of *all* solutions of the analogues problems to the one considered throughout Chapter 2, but in the field of formal Laurent series (compare with 1.23)

$$\left|\alpha - \frac{P}{Q}\right| < \frac{f(\deg Q)}{|Q|^2} \tag{3.1}$$

where P, Q are polynomials with $Q \neq 0$, f is a suitable function, and α is randomly chosen in H (according to the probability measure h).

As in the classical theory, we have the following zero-one law which has to be compared with Khintchine's Theorem 1.3.1 in Section 1.3.1

Theorem 3.1.1. *Let f be a positive, non-increasing, real-valued function defined on the non-negative real numbers. Then, the inequality (3.1) possesses a.s. infinitely many solutions or a.s. finitely many solutions according to whether the following series*

$$\sum_{k=0}^{\infty} f(k) \tag{3.2}$$

diverges or converges.

Next, we want to take a closer look on the case where (3.1) possesses a.s. infinitely many solutions. Therefore, suppose that f is a positive real-valued function defined on the non-negative real numbers satisfying (1.25) and (2.1). Furthmore, we put

$$F(n) := \sum_{k=0}^{n} q^{\lceil \log_q f(k) \rceil}$$

and define the following sequence of random variables

$$T_n(\alpha) := \#\{\langle P, Q\rangle | 0 \leq \deg Q \leq n, P/Q \text{ is a solution of (3.1)}\}.$$

As it was mentioned in Section 1.3.2, in the classical case, LeVeque conjectured a central limit theorem for this sequence of random variables and this conjecture was settled in Section 2.1. We have the following result that proves LeVeque's conjecture in the Laurent series case, too.

Theorem 3.1.2. *We have*

$$\lim_{n\longrightarrow\infty} h\left[T_n < \sigma_1 F(n) + (\sigma_2 F(n) \log F(n))^{1/2}\right] = \Phi(\omega)$$

where

$$\sigma_1 = q-1, \quad \sigma_2 = \frac{(q+1)(q-1)^2}{2q\log q}.$$

We will see that the proof of the above theorem in the Laurent series case is much easier than in the classical case because we only have to apply the theory of independent random variables in contrast to the real case where the theory of mixing sequences of random variables was needed.

Also LeVeque's Theorem 1.3.3 in Section 1.3.2 possesses an analogue in the field of formal Laurent series. Furthermore, we can add without difficulties a law of the iterated logarithm that will follow from Kolmogorov's law of the iterated logarithm for independent sequences of random variables. In order to state the result, now suppose f to be a positive, real-valued function defined on the non-negative real numbers satisfying (1.25) and (1.26). Furthermore, we again put

$$F(n) := \sum_{k=0}^{n} q^{\lceil \log_q f(k) \rceil}$$

and define a sequence of random variables as

$$T_n^{(d)}(\alpha) := \#\{\langle P, Q\rangle | 0 \le \deg Q \le n, \deg(P,Q) \le d, \\ P/Q \text{ is a solution of } (3.1)\}.$$

where $d \geq 1$ is a fixed integer. Then, we have the following result

Theorem 3.1.3. *We have*

$$\lim_{n\longrightarrow\infty} h\left[T_n^{(d)} < \sigma F(n) + \omega(\tau^2 F(n))^{1/2}\right] = \Phi(\omega)$$

and

$$\limsup_{n\longrightarrow\infty} \frac{|T_n^{(d)} - \sigma F(n)|}{\sqrt{2\tau^2 F(n) \log\log F(n)}} = 1$$

where

$$\sigma = (q-1)(1-q^{-d-1}), \quad \tau^2 = \frac{(q-1)^2(q+1)}{q}(d+1).$$

3.1.2 Khintchine's Theorem for Formal Laurent Series

In order to proof Theorem 3.1.1, we start with two lemmas

Lemma 3.1.1. *Let f be a positive function defined on the positive integers and consider the inequality*

$$|A_k(\alpha)| > f(k). \tag{3.3}$$

Then, (3.3) holds for infinitely many k a.s. or (3.3) holds for finitely many k a.s. depending on whether the following series

$$\sum_{k=1}^{\infty} \frac{1}{f(k)}$$

diverges or converges.

Proof. This follows in a straightforward manner from Corollary 1.4.2 and Corollary 1.4.3 by applying the classical Borel-Cantelli lemma. □

Lemma 3.1.2. *Let $(c_k)_{k \geq 0}$ denote a sequence of positive real numbers satisfying $\sum_{k=0}^{\infty} c_k = \infty$. Then, we have a.s. for infinitely many k*

$$\left| \alpha - \frac{P_k}{Q_k} \right| < \frac{c_k}{|Q_k|^2}.$$

Proof. Observe, by (3) and (4) of Lemma 2.2.5

$$|Q_k||Q_k\alpha - P_k| = \frac{|Q_k|}{|Q_{k+1}|} = \frac{1}{|A_{k+1}|}$$

and therefore, the desired result follows immediatly by applying the previous lemma. □

Now, we can start with the proof of Khintchine's Theorem for formal Laurent series

Proof of Theorem 3.1.1. First, we assume that (3.2) converges. We define sets B_k for $k \geq 0$ by

$$B_k = \{\alpha \in H | (3.1) \text{ possesses a solution } P, Q \text{ with } \deg Q = k, \deg P < \deg Q\}$$

and sets $B_{P,Q}$ for polynomials $P, Q \in \mathbb{F}_q[T]$ with $Q \neq 0, \deg P < \deg Q$ by

$$B_{P,Q} = \{\alpha \in H | (3.1) \text{ possesses } P, Q \text{ as solution}\}.$$

From the definition, it is plain that

$$B_k = \bigcup_{\deg Q = k, \deg P < \deg Q} B_{P,Q}$$

and applying Lemma 1.4.2 gives

$$h(B_{P,Q}) = \frac{q^{\lceil \log_q f(\deg Q)\rceil}}{|Q|^2} \leq q\frac{f(\deg Q)}{|Q|^2}.$$

Hence,

$$\sum_{k=0}^{\infty} h(B_k) \leq q(q-1)\sum_{k=0}^{\infty} q^{2k}\frac{f(k)}{q^{2k}} = q(q-1)\sum_{k=0}^{\infty} f(k) < \infty$$

and the first part of our result follows by the classical Borel-Cantelli lemma.

Next, we assume that (3.2) diverges. Then, it is immediate that the following series

$$\sum_{k=0}^{\infty} f(tk) \tag{3.4}$$

diverges as well for all integers $t \geq 0$. Furthermore, Theorem 1.2.5 implies that for t suitable large and fixed, we have a.s.

$$\deg Q_k < tk \tag{3.5}$$

provided that k is large enough. Now, from (3.4) and Lemma 3.1.2, we get

$$\left|\alpha - \frac{P_k}{Q_k}\right| < \frac{f(tk)}{|Q_k|^2}$$

and (3.5) together with the monotony of f implies the desired result. □

3.1.3 LeVeque's Problem in the Laurent Series Case

The preparation steps for the proof of Theorem 3.1.2 are similar to the ones performed in Section 2.1.2. First, define a sequence of random variables by

$$U_k(\alpha) := \#\{0 \leq \deg C | |C|^2 < |A_{k+1}| f(\log CQ_k)\}.$$

This sequence can be used to approximate $(T_n)_{n\geq 0}$

Lemma 3.1.3. *We have*

$$\sum_{\deg Q_{k+1}\leq n} U_k(\alpha) \leq T_n(\alpha) \leq \sum_{\deg Q_k \leq n} U_k(\alpha). \tag{3.6}$$

Proof. The proof essentially runs along the same lines than the proof of Lemma 2.2.2 of Section 2.2.2.

The only difference is that because of (3) and (4) of Lemma 1.4.1 in Section 1.4.1, we have

$$\left|\alpha - \frac{P_k}{Q_k}\right| = \frac{1}{|Q_k||Q_{k+1}|} = \frac{1}{|Q_k|^2|A_{k+1}|},$$

and consequently

$$\left|\alpha - \frac{P_k}{Q_k}\right| < \frac{f(\deg CQ_k)}{|CQ_k|^2} \Longleftrightarrow |C|^2 < |A_{k+1}| f(\deg CQ_k).$$

□

In Section 2.1.2, the analogues sequence to $(U_k)_{k\geq 0}$ was in turn approximated by using the law of the iterated logarithm for the denominators of the convergents in the continued fraction expansion proved by Gordin and Reznik (see Theorem 1.2.5 in Section 1.2.1). Since, we have Theorem 1.4.2 of Section 1.4.2 instead of it, we can proceed similar as in Section 2.1.2 and approximate once more. Therefore, we set

$$f_1(k) := f((k+1)\log\gamma + \kappa(k+1)^{1-\delta}), \quad f_2(k) := f(k\log\gamma + \kappa k^{1-\delta}),$$

where κ is as in Section 2.1.2 and we put

$$F_i(n) = \sum_{k=0}^{n} q^{\lceil \log_q f_i(k)\rceil}, \quad i = 1, 2.$$

Furthermore, we define a sequence of random variables as

$$V_k^{(i)}(\alpha) := \#\{0 \leq \deg C \,|\, |C|^2 < |A_{k+1}| f_i(k)\}.$$

Then, we have

$$V_k^{(1)}(\alpha) \leq U_k(\alpha) \leq V_k^{(2)}(\alpha)$$

for a subset F of H with $h(F) \geq 1 - \epsilon$ (compare again with Section 2.1.2).

Our next aim is to provide a proof of the central limit theorem for the approximating sequences $(V_k^{(i)})_{k\geq 0}$. Therefore, observe that these sequences of random variables are independent due to Corollary 1.4.3 and consequentely, this aim can easily be achieved by using the theory of independent sequences of random variables.

Lemma 3.1.4. *We have, as $n \longrightarrow \infty$,*

$$h\left[\sum_{k=0}^{n} V_k^{(i)} < qF_i(n) + \omega((q^2-1)/(2\log q)F_i(n)\log F_i(n))^{1/2}\right] \longrightarrow \Phi(\omega). \tag{3.7}$$

Proof. In the following, we are going to supress the dependence on i.

As already mentioned, Corollary 1.4.3 implies that the sequence of random variables under investigation is an independent one. However, it is not possible to deduce (3.7) just by proving Lindeberg's conditions since the second moments of V_k don't exist. Therefore, we need a central limit theorem for independent sequences of random variables without any moment condition and such a result was provided in Section 1.1.1. Here, we are going to apply this theorem.

Therefore, we define

$$c_n = (F(n))^{1/2}(\log F(n))^{1/2-\rho} - 1$$

where $0 < \rho < 1/2$ is a real constant. By using this sequence, we will prove (1.1) and (1.2). First observe

$$V_k(\alpha) = q^l - 1 \quad \Longleftrightarrow \quad \frac{q^{2(l-1)}}{f(k)} < |A_{k+1}| \leq \frac{q^{2l}}{f(k)} \tag{3.8}$$

and hence

$$h[V_k \geq c_n] = h\left(\{\alpha \in H | q^{2(\lceil\log_q(c_n+1)\rceil-1)}/f(k) < |A_{k+1}|\}\right).$$

Applying Corollary 1.4.2 and summing up yields

$$\sum_{k=0}^{n} h[V_k \geq c_n] = q^{-2\lceil\log_q(c_n+1)\rceil+2}F(n) \ll \frac{1}{(\log F(n))^{1-2\rho}}$$

which tends to 0 as n tends to ∞. Therefore, we have proved (1.1).

In order to prove (1.2), observe by (3.8)

$$\begin{aligned}
\mathbf{E}V_k^{(c_n)} &= \sum_{l=1}^{\lceil\log_q(c_n+1)\rceil-1} (q^l-1)h\left(\{\alpha \in H | q^{2(l-1)}/f(k) < |A_{k+1}| \leq q^{2l}/f(k)\}\right) \\
&= q^{\lceil\log_q f(k)\rceil} \sum_{l=1}^{\lceil\log_q(c_n+1)\rceil-1} (q^l-1)(q^{-2(l-1)} - q^{-2l}) \\
&= q^{\lceil\log_q f(k)\rceil}(q-(q+1)q^{-\lceil\log_q(c_n+1)\rceil+1} + q^{-2\lceil\log_q(c_n+1)\rceil+2}).
\end{aligned}$$

A similar computation for the second moment of $V_k^{(c_n)}$ gives

$$\begin{aligned}
\mathbf{E}(V_k^{(c_n)})^2 = q^{\lceil\log_q f(k)\rceil}((q^2-1)\lceil\log_q(c_n+1)\rceil - q(q+2)+ \\
+ 2(q+1)q^{-\lceil\log_q(c_n+1)\rceil+1} - q^{-2\lceil\log_q(c_n+1)\rceil+2}).
\end{aligned}$$

Consequently,

$$\frac{1}{c_n^2}\sum_{k=0}^{n} \mathbf{V}V_k^{(c_n)} = \frac{1}{c_n^2}((q^2-1)/2F(n)\log_q F(n) + \mathrm{O}(F(n)\log\log F(n)))$$

which tends to ∞ as n tends to ∞. This proves (1.2) and therefore, we can apply Theorem 1.1.1 of Section 1.1.1. Thus, we know that the sequence V_k satisfies the central limit theorem if properly normalized.

By Theorem 1.1.1 of Section 1.1.1. the proper normalization is given by (we use the notation of the theorem)

$$a_n^2 = \sum_{k=0}^{n} \mathbf{V} V_k^{(c_n)} \sim (q^2-1)/(2\log q) F(n) \log F(n)$$

and

$$b_n = \sum_{k=0}^{n} \mathbf{E} V_k^{(c_n)} = qF(n) + \mathrm{O}((F(n))^{1/2}).$$

Comparison with (3.7) shows that this is the desired result. □

Now, in order to finish the proof of Theorem 3.1.2, we can proceed as in Section 2.1.4. Therefore, it is sufficient to point out that Lemma 2.1.7 can be replaced by

Lemma 3.1.5. *Let g_1 (resp. g_2) be the inverse function of $(k+1)\log\gamma + \kappa(k+1)^{1-\delta}$ (resp. $k\log\gamma - \kappa k^{1-\delta}$). Then we have*

$$F_i(g_i(n)) = \frac{1}{\log\gamma} F(n) + O(F(n)^{1/2}).$$

3.1.4 Proof of Theorem 3.1.3

The preparation steps for the proof of Theorem 3.1.3 are nearly the same within the last section. The only difference is that we have to replace the sequence $(U_k)_{k\geq 0}$ of the last section by

$$U_k(\alpha) := \#\{0 \leq \deg C \leq d \,|\, |C|^2 < |A_{k+1}| f(\log CQ_k)\}.$$

and the sequences $(V_k^{(i)})_{k\geq 0}$ by

$$V_k^{(i)}(\alpha) := \#\{0 \leq \deg C \leq d \,|\, |C|^2 < |A_{k+1}| f_i(k)\}.$$

Next, we are going to compute the mean value and the variance of the sequences $(V_k^{(i)})_{k\geq 0}$. Because of Corollary 1.4.2 this is an easy task

Lemma 3.1.6. *Set*

$$F_i(n) = \sum_{k=0}^{n} q^{\lceil \log_q f_i(k) \rceil}.$$

Then, we have

$$\mathbf{E} \sum_{k=0}^{n} V_k^{(i)} = \frac{q}{q-1} \sigma F_i(n) \tag{3.9}$$

and

$$\mathbf{V} \sum_{k=0}^{n} V_k^{(i)} \sim \frac{q}{q-1} \tau^2 F_i(n). \tag{3.10}$$

Proof. Since there is no difference between $i = 1$ and $i = 2$, we assume w.l.o.g. $i = 1$. Furthermore, in the following we are going to suppress the dependence on i.

First, we write

$$V_k = \sum_{0 \le \deg C \le d} V_{k,C}$$

where

$$V_{k,C}(\alpha) = \begin{cases} 1 & \text{if } |\xi_k| < f(k)/|C|^2, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, Corollary 1.4.1 implies

$$\mathbf{E}V_k = \sum_{0 \le \deg C \le d} \mathbf{E}V_{k,C} = q^{\lceil \log_q f(k) \rceil} \sum_{0 \le \deg C \le d} \frac{1}{|C|^2}. \tag{3.11}$$

Finally, summing up and a little bit straightforward calculation immediatley gives (3.9).

For the proof of (3.10) observe

$$\mathbf{E}V_k^2 = \sum_{l=0}^{d} (q^{2l}(q-1)^2 + 2q^{2l}(q-1)) q^{\lceil \log_q f(k) \rceil} q^{-2l} \tag{3.12}$$

where again Corollary 1.4.1 was used. Furthermore, (3.11) gives

$$(\mathbf{E}V_k)^2 \ll f(k)^2. \tag{3.13}$$

By (3.12) and (3.13) together with summing up and a little bit simple calculation the relation (3.10) is obvious. □

Since Corollary 1.4.3 implies that the sequences $(V_k^{(i)})_{k \ge 0}$ are independent sequences of random variables, it is quite easy to obtain a central limit theorem resp. law of the iterated logarithm just by making use of the theory of independent sequences of random variables

Lemma 3.1.7. *Set*

$$F_i(n) = \sum_{k=0}^{n} q^{\lceil \log_q f_i(k) \rceil}.$$

Then, we have

$$\lim_{n \longrightarrow \infty} h \left[\sum_{k=0}^{n} V_k^{(i)} < q/(q-1)\sigma F_i(n) + \omega (q/(q-1)\tau^2 F_i(n))^{1/2} \right] = \Phi(\omega) \tag{3.14}$$

and

$$\limsup_{n \longrightarrow \infty} \frac{\left| \sum_{k=0}^{n} V_k^{(i)} - q/(q-1)\sigma F_i(n) \right|}{\sqrt{2q/(q-1)\tau^2 F_i(n) \log\log F_i(n)}} = 1. \tag{3.15}$$

Proof. As in the proof of the last lemma, we are going to suppress the dependence on i. First, we normalize the sequence $(V_k)_{k\geq 0}$

$$\eta_k = V_k - \mathbf{E}V_k.$$

Observe, that the sequence $(\eta_k)_{k\geq 0}$ is uniformly bounded because of the definition of V_k on the one hand and (3.11) on the other hand. Furthermore, (3.10) implies

$$\mathbf{V}\sum_{k=0}^{n}\eta_k \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty.$$

Therefore, it is plain that Lindeberg's condition and the conditions in Kologorov's law of the iterated logarithm (see for instance Theorem 1 of Chapter X in [22]) are satisfied for the sequence $(\eta_k)_{k\geq 0}$ and since this sequence is independent by Corollary 1.4.3, we immediately get (3.14) and (3.15). $\square$

As in the last section, we finish the proof of Theorem 3.1.3 by applying the method introduced in Section 2.1.4. Here, it is sufficient to observe that Lemma 2.2.9 can be replaced by

Lemma 3.1.8. *Let g_1 (resp. g_2) be the inverse function of $(k+1)\log\gamma + \kappa(k+1)^{1-\delta}$ (resp. $k\log\gamma - \kappa k^{1-\delta}$). Then we have*

$$F_i(g_i(n)) = \frac{1}{\log\gamma}F(n) + O(F(n)^{1/2-\delta_2}).$$

3.2 Counting Solutions contained in Arithmetic Progressions - The Mixing Case

In this section, we are proving (amongst other results) analogues in the field of formal Laurent series of the theorems obtained in Chapter 2. Thereby, we improve the results of the last section. While in the real case, there is no difference in the methods for counting all solutions and counting solutions that have denominators contained in an arithmetic progression, the methods in the formal Laurent series case differ a lot. This is because we have to pass from the theory of independent sequences of random variables to the theory of mixing sequences of random variables.

All results of Chapter 2 rest on Theorem 1.2.6 of Szüsz and there exists no analogue of this theorem in literature for the formal Laurent series case. Therefore, we start by providing a suitable analogue that will be the key result for all further investigations (for details see Section 3.2.2). Then Section 3.2.3, 3.2.4, and 3.2.5 are used in order to give several applictions of this analogue of Szüsz's Theorem to the theory of diophantine approximation and to the theory of continued fraction expansion as well. Since there are many similarities to the methods used in Chapter 2, we will mainly focus on differences. In the next section, we will start by stating the results.

3.2.1 Statement of the Results

The following result is an analogue of Theorem 1.2.6 in Section 1.2.2

Theorem 3.2.1. *For $x \in [0,1], S, T, R \in \mathbb{F}_q[T]$, and $\deg R \geq 0$ define*

$$m_k(S,T,x) := h\left(\{\alpha \in H | Q_{k-1} \equiv S\ (R), Q_k \equiv T\ (R), |\xi_k| < x\}\right).$$

Then, we have

$$m_k(S,T,x) = \begin{cases} \frac{1}{C(R)} q^{\lceil \log_q x \rceil}(1 + O(\rho^k)) & \text{if } (S,T,R) = 1, \\ 0 & \text{if } (S,T,R) \neq 1, \end{cases}$$

where $C(R) = |R|^2 \prod_{P|R}\left(1 - \frac{1}{|P|^2}\right)$ (the product is extended over all monic, irreducible polynomials P dividing R), $\rho < 1$ is a constant, and the constant implied in the error term depends only an R.

Furthermore, we can use the technique of proof of this theorem to obtain the following asymptotic independence result

Theorem 3.2.2. *For $x \in [0,1], A_1, \ldots, A_l, S, T, R \in \mathbb{F}_q[T], \deg R \geq 0$, and $1 \leq \deg A_i, 1 \leq i \leq l$ define*

$$\begin{aligned} m_k(A_1, \ldots, A_l; S, T, x) := \{\alpha \in H | A_1(\alpha) = A_1, \ldots, A_l(\alpha) = A_l, \\ Q_{k-1} \equiv S\ (R), Q_k \equiv T\ (R), |\xi_k| < x\} \end{aligned}$$

and

$$m(A_1, \ldots, A_l) := \{\alpha \in H | A_1(\alpha) = A_1, \ldots, A_l(\alpha) = A_l\}$$

where $A_i(\alpha)$ is the i-th partial quotient in the continued fraction expansion of α. Then, we have

$$\frac{m_k(A_1, \ldots, A_l; S, T, x)}{m(A_1, \ldots, A_l)} = \begin{cases} \frac{1}{C(R)} q^{\lceil \log_q x \rceil}(1 + O(\rho^{k-l})) & \text{if } (S,T,R) = 1, \\ & \text{if } (S,T,R) \neq 1, \end{cases}$$

where $C(R), \rho$ are as in Theorem 3.2.1, and the implied constant again depends only on R.

As in the classical theory, these two results have several applications. First, we concentrate on the metric theory of continued fraction expansion.

We give a common extension of Theorem 1.2.7 in Section 1.2.2 (or more precisely of the analogue of this result in the Laurent series case which is straightforward to prove) and a recent result of Harman and Wong (see Theorem 1 in [12]). Therefore, we define

Definition 3.2.1. Let $B_1, \ldots, B_l, R \in \mathbb{F}_q[T]$ and $\deg R \geq 0$ be given. We call the l-tuple $B_1, \ldots, B_l$ acceptable mod R if

$$(B_i, B_{i+1}) = 1, \quad 1 \leq i < l$$

and

$$(R, B_{i-1}) | B_i - B_{i-2}, \quad 3 \leq i \leq l.$$

A l-tuple $B_1, \ldots, B_l$ which is not acceptable mod R is said to be prohibited mod R.

Using this notation, we can show the following theorem

Theorem 3.2.3. *Let $f(C_1, \ldots, C_l, A)$ be a function where $C_1, \ldots, C_l, A \in \mathbb{F}_q[T], \deg A \geq 1$, and $l \geq 2$. Furthermore, assume that f is periodic mod R in the first l coordinates and satisfies*

$$|f(C_1, \ldots, C_l, A)| \ll |A|^{1/2-\delta}$$

where $\delta > 0$ is a real constant. Denote by $(k_i)_{i\geq 1}$ an increasing sequence of positive integers. Then, we have

$$\sum_{i \leq n} f(Q_{k_i}, \ldots, Q_{k_i+l-1}, A_{k_i+l}) = C(f)n + O\left(n^{1/2}(\log n)^{3/2+\epsilon}\right) \quad a.s.$$

for all $\epsilon > 0$. Here

$$C(f) = \sum_{\substack{A \in \mathbb{F}_q[T], \deg A \geq 1 \\ B_1, \ldots, B_l \text{ acceptable}}} f(B_1, \ldots, B_l, A) C(B_1, \ldots, B_l) q^{-2 \deg A}$$

and

$$C(B_1, \ldots, B_l) = \frac{1}{C(R)} \prod_{j=3}^{l} \sum_{C \in L_j} q^{-2 \deg C}$$

where

$$L_j = \{A \in \mathbb{F}_q[T] | AB_{j-1} \equiv B_j - B_{j-2} \ (R)\}.$$

This result entails the following interesting consequence

Corollary 3.2.1. *Let $S, R \in \mathbb{F}_q[T], \deg R \geq 0$ be given. Assume that $(k_i)_{i\geq 1}$ is an increasing sequence of positive integers. Then, we have a.s.*

$$Q_{k_i} \equiv S \ (R)$$

for infinitely many i.

Remark 3.2.1. This consequence has to be compared with a remark in [12] were the special case, that $(k_i)_{i\geq 1}$ is an arithmetic progression was considered. Indeed, a similar result is also true in the classical case (the method of the proof can be used without difficulties).

Next, we show applications to the metric theory of diophantine approximation.

Therefore, let f be a positive, real-valued function defined on the non-negative real numbers satisfying (1.25) and (2.1). Furthermore, fix polynomials S, R with $\deg R \geq 0$ and define the following sequence of random variables

$$X_n(\alpha) := \#\{\langle P, Q\rangle | 0 \leq \deg Q \leq n, Q \equiv S \quad (R), P/Q \text{ is a solution of } (3.1)\}.$$

In Section 3.1 of this chapter, we already obtained a central limit theorem for the sequence of random variables arising from counting all solutions of (3.1). This result generalizes to the following one (compare with Theorem 2.1.1 of Section 2.1 in the classical case)

Theorem 3.2.4. *Set*

$$F(n) := \sum_{k=0}^{n} q^{\lceil \log_q f(k) \rceil}.$$

Then, we have

$$\lim_{n \longrightarrow \infty} h\left[X_n \leq \sigma_1 F(n) + \omega(\sigma_2 F(n) \log F(n))^{1/2}\right] = \Phi(\omega).$$

Here,

$$\sigma_1 = \frac{q-1}{|R|}, \quad \sigma_2 = \frac{(q+1)(q-1)^2 |(S,R)| \varphi(R)}{2q \log q |R| C(S,R)},$$

and

$$C(S,R) = |R|^2 \prod_{P|R} \left(1 - \frac{1}{|P|}\right) \prod_{P|\frac{R}{(S,R)}} \left(1 + \frac{1}{|P|}\right),$$

where the products are extended over all monic, irreducible polynomials satisfying the desired properties.

As considered in Section 2.2., the situation is getting much easier if we replace (2.1) by (1.26) on the one hand and X_n by the following sequence of random variables on the other hand

$$\begin{aligned} X_n^{(d)}(\alpha) := \#\{\langle P, Q\rangle | 0 \leq \deg Q \leq n, \deg(P,Q) \leq d, \\ Q \equiv S \ (R), P/Q \text{ is a solution of } (3.1)\} \end{aligned}$$

where $d \geq 1$ is a fixed integer.

Then, Theorem 2.2.1 and Theorem 2.2.2 of Section 2.2 carry over in the Laurent series case. Therefore, we put

$$F(n) = \sum_{k=0}^{n} q^{\lceil \log_q f(k) \rceil}$$

and

$$\sigma = \sigma_d = \frac{q-1}{q} \sum_{\deg D < \deg R} \sum_{\substack{0 \le \deg C \le d \\ CD \equiv S\ (R)}} \frac{K(D)}{|C|^2},$$

$$\tau^2 = \tau_d^2 = \frac{q-1}{q} \sum_{\deg D < \deg R} \sum_{0 \le l \le d} k_{l,D} q^{-2l},$$

where $k_{l,D}$ is the number of pairs $\langle C_1, C_2 \rangle$ with C_1, C_2 solutions of $CD \equiv S\ (R)$, $\deg C_1, \deg C_2 \le l$ and either $\deg C_1 = l$ or $\deg C_2 = l$. Furthermore, set

$$n_t = \begin{cases} \max\{n | \tau^2 F(n) \le t\} & \text{if } t \ge \tau^2 F(0), \\ 0 & \text{otherwise,} \end{cases}$$

where $t \ge 0$.

Now, consider the product probability space $(H, \mathcal{H}, h) \times ([0,1], \mathcal{B}, \lambda)$. Using the above notation, we define on this probability space a stochastic process Y by setting for $\alpha \in H$ and $x \in [0,1]$

$$X(t) = X(t; \alpha, x) = X_{n_t}^{(d)}(\alpha) - \sigma F(n_t)$$

This definition has to be compared with the corresponding process defined in Section 2.2.1.

Theorem 3.2.5. *There exists a sequence $(Y_i(\alpha, x))_{i \ge 1}$ of independent, standard normal $\mathcal{N}(0,1)$ random variables defined on the above probability space such that, as $n \longrightarrow \infty$,*

$$\Big| X(n) - \sum_{i \le n} Y_i \Big| = o((n \log\log n)^{1/2}) \ a.s.$$

and

$$(h \times \lambda) \left[\frac{1}{\sqrt{n}} \max_{k \le n} \Big| X(k) - \sum_{i \le k} Y_i \Big| \ge \epsilon \right] \longrightarrow 0$$

for all $\epsilon > 0$.

As in Section 2.2, we can even prove more if f satisfies an additional assumptions

Theorem 3.2.6. *Let f satisfy (1.25), (1.26), and (2.39). Then, there exists a sequence $(Y_i(\alpha, x))_{i \ge 1}$ of independent, standard normal $\mathcal{N}(0,1)$ random variables defined on the above probability space such that, as $n \longrightarrow \infty$,*

$$X(n) - \sum_{i \le n} Y_i \ll n^{1/2 - \lambda} \quad a.s.$$

where $0 < \lambda < 1/2$ is a real constant.

Remark 3.2.2. As it was pointed out in Section 2.2 these theorems entail several consequences, for instance, a functional central limit theorem and a Strassen's type version of the law of the iterated logarithm. Under the stronger assumptions of the second theorem, we even obtain a functional law of the iterated logarithm for the maximum.

Remark 3.2.3. Theorem 3.2.5 as well as Theorem 3.2.6 contain Theorem 3.1.3 of Section 3.1. However, as already mentioned the situation is now totally different since the involved random variables are mixing and not independent any more.

3.2.2 Szüsz's Theorem for Formal Laurent Series

We need a technical lemma for the proofs of Theorem 3.2.1 and Theorem 3.2.2. But before we can state it, we have to indroduce notation. Therefore, define for given polynomials $S, T, R, C_1, C_2, C_3, C_4 \in \mathbb{F}_q[T]$ with $\deg R \geq 0$, consecutive

$$\begin{array}{rcll} U_1(C_1) & \equiv & T - C_1 S & (R), \\ U_2(C_2) & \equiv & S - C_2 U_1(C_1) & (R), \\ U_3(C_3) & \equiv & U_1(C_1) - C_3 U_2(C_2) & (R), \\ U_4(C_4) & \equiv & U_2(C_2) - C_4 U_3(C_3) & (R). \end{array}$$

As in [30], the following elementary lemma is straightfoward to prove

Lemma 3.2.1. *Let $S, T, \bar{S}, \bar{T}, R \in \mathbb{F}_q[T]$ be given polynomials with $\deg R \geq 0$ and $(S, T, R) = 1, (\bar{S}, \bar{T}, R) = 1$. There are polynomials $C_1, C_2, C_3, C_4 \in \mathbb{F}_q[T]$ such that*

$$\begin{array}{rcl} U_3(C_3) & \equiv & \bar{T} \ (R), \\ U_4(C_4) & \equiv & \bar{S} \ (R). \end{array}$$

Now, we can start proving the Theorems.

Proof of Theorem 3.2.1. Because of the fact that denominators of two consecutive convergents have to be coprime, it is clear that $m_k(S, T, x) = 0$ for all polynomials with $(S, T, R) \neq 1$. Therefore, we can concentrate on polynomials S, T with $(S, T, R) = 1$.

Now, Corollary 1.4.1 and Corollary 1.4.3 of Section 1.4.2 imply

$$\begin{aligned} m_k(S, T, x) &= h(\{\alpha \in H | Q_{k-1} \equiv S \ (R), Q_k \equiv T \ (R)\}) \cdot \\ &\qquad \cdot h\left(\{\alpha \in H || \xi_k| < x\}\right) \\ &= h(\{\alpha \in H | Q_{k-1} \equiv S \ (R), Q_k \equiv T \ (R)\} q^{\lceil \log_q x \rceil}), \end{aligned}$$

and so, it is sufficent to consider

$$m_k(S, T) := h\left(\{\alpha \in H | Q_{k-1} \equiv S \ (R), Q_k \equiv T \ (R)\}\right).$$

Using Corollary 1.4.2 and once more Corollory 1.4.3 of Section 1.4.2, we obtain

$$\begin{aligned} m_k(S,T) = & \sum_{C_1\in\mathbb{F}_q[T],\deg C_1\geq 1} h(\{\alpha\in H|Q_{k-2}\equiv U_1(C_1)\ (R), \\ & \qquad\qquad Q_{k-1}\equiv S\ (R), A_k=C_1\}) \qquad (3.16) \\ = & \sum_{C_1\in\mathbb{F}_q[T],\deg C_1\geq 1} m_{k-1}(U_1(C_1),S)q^{-2\deg C_1} \end{aligned}$$

and by applying the last equality three times we get

$$m_k(S,T) = \sum_{C_1,C_2,C_3,C_4} m_{k-4}(U_4(C_4),U_3(C_3))q^{-2(\deg C_1+\deg C_2+\deg C_3+\deg C_4)}$$

where the sum runs over all polynomials of degree at least 1.

Next, define

$$\begin{aligned} m_k &= \min_{(S,T,R)=1,\deg S,\deg T<\deg R} m_k(S,T), \\ M_k &= \max_{(S,T,R)=1,\deg S,\deg T<\deg R} m_k(S,T), \end{aligned}$$

and observe

$$\begin{aligned} m_k(S,T) = & \sum_{C_1\in\mathbb{F}_q[T],\deg C_1\geq 1} m_{k-1}(U_1(C_1),S)q^{-2\deg C_1} \\ & \geq m_{k-1}\sum_{C_1\in\mathbb{F}_q[T],\deg C_1\geq 1} q^{-2\deg C_1} = m_{k-1} \end{aligned}$$

for all polynomials S,T with $(S,T,R)=1$. Consequently, the sequence m_k is a non-decreasing sequence of real numbers.

Since

$$\sum_{C_1,C_2,C_3,C_4} q^{-2(\deg C_1+\deg C_2+\deg C_3+\deg C_4)} = 1,$$

where the sum runs over all polynomials with degree at least 1, we have

$$\begin{aligned} m_k(S,T) - m_k \leq & \sum_{C_1,C_2,C_3,C_4} (m_{k-4}(U_4(C_4),U_3(C_3)) - m_{k-4})\cdot \\ & \qquad \cdot q^{-2(\deg C_1+\deg C_2+\deg C_3+\deg C_4)}. \end{aligned}$$

Let $\bar{S},\bar{T}$ be polynomial with $(\bar{S},\bar{T},R)=1,\deg\bar{S},\deg\bar{T}<\deg R$ satisfying $m_{k-4}=m_k(\bar{S},\bar{T})$. According to Lemma 3.2.1, we have polynomials D_1,D_2,D_3,D_4 such that $U_3(D_3)=\bar{T},U_4(D_4)=\bar{S}$. Hence

$$\begin{aligned} m_k(S,T) - m_k \leq & {\sum}^*(m_{k-4}(U_4(C_4),U_3(C_3)) - m_{k-4})\cdot \\ & \qquad \cdot q^{-2(\deg C_1+\deg C_2+\deg C_3+\deg C_4)} \end{aligned}$$

where the sum runs over all polynomials C_1, C_2, C_3, C_4 with degree at least 1 and not satisfying $C_1 \equiv D_1 \ (R), C_2 \equiv D_2 \ (R), C_3 \equiv D_3 \ (R), C_4 \equiv D_4 \ (R)$. Thus,

$$m_k(S,T) - m_k \leq (M_{k-4} - m_{k-4}) \sum{}^{*} q^{-2(\deg C_1 + \deg C_2 + \deg C_3 + \deg C_4)}$$

and an easy calculation gives

$$\sum{}^{*} q^{-2(\deg C_1 + \deg C_2 + \deg C_3 + \deg C_4)} \leq \bar{\rho}$$

where $\bar{\rho} < 1$ is a constant which depends only on R. Therefore

$$M_k - m_k \leq (M_{k-4} - m_{k-4})\bar{\rho}$$

which implies $M_k - m_k = \mathrm{O}(\rho^k)$ for a suitable constant ρ.

Consequently, we have for fixed polynomials S, T with $(S,T,R) = 1$

$$m_k(\bar{S}, \bar{T}) = m_k(S,T) + \mathrm{O}(\rho^k)$$

for all polynomials $\bar{S}, \bar{T}$ with $(\bar{S}, \bar{T}, R) = 1$. This implies

$$\begin{aligned} 1 &= \sum_{(\bar{S},\bar{T},R)=1, \deg \bar{S}, \deg \bar{T} < \deg R} m_k(\bar{S}, \bar{T}) \\ &= (m_k(S,T) + \mathrm{O}(\rho^k)) \sum_{(\bar{S},\bar{T},R)=1, \deg \bar{S}, \deg \bar{T} < \deg R} 1. \end{aligned}$$

and since

$$\sum_{(\bar{S},\bar{T},R)=1, \deg \bar{S}, \deg \bar{T} < \deg R} 1 = C(R)$$

we get

$$m_k(S,T) = \frac{1}{C(R)} + \mathrm{O}(\rho^k)$$

which gives the desired result. $\square$

Proof of Theorem 3.2.2. Define

$$\begin{aligned} m_k(A_1, \ldots, A_l; S, T) := h(\{\alpha \in H | A_1(\alpha) = A_1, \ldots, A_l(\alpha) = A_l, \\ Q_{k-1} \equiv S \ (R), Q_k \equiv T \ (R)\}). \end{aligned}$$

As in the proof of Theorem 3.2.1, it suffices to concentrate on the above quantity on the one hand and to consider only polynomials S, R with $(S,T,R) = 1$ on the other hand. Next, observe that

$$\frac{m_k(A_1, \ldots, A_l; S, T)}{m(A_1, \ldots, A_l)}$$

satisfies recurrence (3.16). Furthermore, we have the right norming. Therefore, we get as in the proof of Theorem 3.2.1

$$\frac{m_k(A_1,\dots,A_l;S,T)}{m(A_1,\dots,A_l)} = \frac{1}{C(R)}(1+\mathrm{O}(\rho^{k-l}))$$

(especially, notice that the implied constant doesn't depend on $A_1,\dots,A_l$) which immediatly gives the result. $\square$

3.2.3 A Result in Metric Theory of Continued Fraction

First, we need a little notation

Definition 3.2.2. Let $B_1,\dots,B_l$ be an acceptable l-tuple mod R and $A \in \mathbb{F}_q[T]$ be a polynomial with $\deg A \geq 1$. Then, we put

$$T_i(B_1,\dots,B_l;A) := \{\alpha \in H | Q_i \equiv B_1\ (R),\dots,Q_{i+l-1} \equiv B_l\ (R), A_{i+l} = A\}.$$

Next, we collect a few simple lemmas

Lemma 3.2.2. *We have*

$$h(T_i(B_1,\dots,B_l;A)) = C(B_1,\dots,B_l)q^{-2\deg A}(1+O(\rho^i)).$$

Here

$$C(B_1,\dots,B_l) = \frac{1}{C(R)}\prod_{j=3}^{l}\sum_{C\in L_j} q^{-2\deg C}$$

where

$$L_j = \{A \in \mathbb{F}_q[T] | AB_{j-1} \equiv B_j - B_{j-2}\ (R)\}.$$

Proof. Observe that

$$T_i(B_1,\dots,B_l;A) = \bigcup_{C_3\in L_3,\dots,C_l\in L_l} \{\alpha \in H | Q_i \equiv B_1\ (R), Q_{i+1} \equiv B_2\ (R),$$
$$A_{i+2} = C_3,\dots,A_{i+l-1} = C_l, A_l = A\}$$

and the result is easily obtained by using Theorem 3.2.1 and Corollary 1.4.2 resp. Corollary 1.4.3 of Section 1.4.2. $\square$

Lemma 3.2.3. *We have*

$$h(T_i(B_1,\dots,B_l;A_1) \cap T_j(C_1,\dots,C_l;A_2))$$
$$= h(T_i(B_1,\dots,B_l;A_1))h(T_j(C_1,\dots,C_l;A_2))(1+O(\rho^{j-i}))$$

for $j \geq i+l-1$.

Proof. Notice that

$$
\begin{aligned}
&T_i(B_1,\dots,B_l;A_1)\cap T_j(C_1,\dots,C_l;A_2)= \\
&\quad\bigcup_{E_1,\dots,E_{i+l-1}}\ \bigcup_{D_3,\dots,D_l}\{\alpha\in H|A_1=E_1,\dots,A_{i+l-1}=E_{i+l-1},A_{i+l}=A_1, \\
&\qquad\qquad Q_j\equiv C_1\ (R),Q_{j+1}\equiv C_2\ (R),A_{j+2}=D_3,\dots,A_{j+l-1}=D_l, \\
&\qquad\qquad A_{j+l}=A_2\}
\end{aligned}
$$

where the first joint runs over all polynomials $E_1,\dots,E_{i+l-1}$ with degree at least 1 such that the denominators of the convergents of $[0,E_1,\dots,E_{i+l-1}]$ satisfy $Q_i\equiv B_1\ (R),\dots,Q_{i+l-1}\equiv B_l\ (R)$ and the second joint runs over all polynomials $D_3,\dots,D_l$ with $D_i\in L_j$ where

$$L_j=\{A\in\mathbb{F}_q[T]|AC_{j-1}\equiv C_j-C_{j-2}\ (R)\}.$$

Applying Theorem 3.2.2, Corollary 1.4.2 and Corollary 1.4.3 of Section 1.4.2, and the last Lemma immediately gives the result. $\square$

The proof of Theorem 3.2.3 follows from these two Lemmas.

Proof of Theorem 3.2.3. Set $X_i := f(Q_{k_i},\dots,Q_{k_{i+l-1}},A_{k_{i+l}})$ and consider

$$
\begin{aligned}
\mathbf{E}X_i &= \sum_{\substack{A\in\mathbb{F}_q[T],\deg A\geq 1\\ B_1,\dots,B_l \text{ acceptable}}} f(B_1,\dots,B_l,A)h(T_{k_i}(B_1,\dots,B_l,A)) \\
&=(1+\mathrm{O}(\rho^{k_i}))C(f).
\end{aligned}
$$

Therefore, we have

$$\mathbf{E}\sum_{i\leq n}X_i=C(f)n+\mathrm{O}(1). \tag{3.17}$$

Next, we estimate

$$\mathbf{E}|X_i|=\sum_{\substack{A\in\mathbb{F}_q[T],\deg A\geq 1\\ B_1,\dots,B_l \text{ acceptable}}}|f(B_1,\dots,B_l,A)|h(T_{k_i}(B_1,\dots,B_l,A))\ll 1$$

and it is clear that we have $\mathbf{E}(X_i)^2\ll 1$ as well. Because of Lemma 3.2.3, we can apply Lemma 1.1.1 of Section 1.1.2 in order to estimate the covariance

$$\mathrm{Cov}(X_{i_1},X_{i_2})\ll\rho^{i_2-i_1}\mathbf{E}|X_{i_1}|\mathbf{E}|X_{i_2}|\ll\rho^{i_2-i_1} \tag{3.18}$$

for $i_2\geq i_1+l-1$. Furthermore, notice

$$
\begin{aligned}
|\mathrm{Cov}(X_{i_1},X_{i_2})| &\leq\mathbf{E}|X_{i_1}X_{i_2}|+\mathbf{E}|X_{i_1}|\mathbf{E}|X_{i_2}| \\
&\ll\mathbf{E}^{1/2}(X_{i_1})^2\mathbf{E}^{1/2}(X_{i_2})^2+1\ll 1
\end{aligned}
\tag{3.19}
$$

for $i_2 > i_1$. Next, consider

$$\mathbf{V} \sum_{i=m+1}^{m+n} X_i = \sum_{i=m+1}^{m+n} \mathbf{V} X_i + 2 \sum_{m+1 \le i_1 < i_2 \le m+n} \mathrm{Cov}(X_{i_1}, X_{i_2}) \tag{3.20}$$

and brake the last sum into two parts $\sum = \sum' + \sum''$ according to whether $i_2 \ge i_1 + l - 1$ or not. Because of (3.18) and (3.19), both parts are bounded by n. Trivially, the first sum on the right hand side of (3.20) is also bounded by n and so, we finally get

$$\mathbf{V} \sum_{i=m+1}^{m+n} X_i \ll n.$$

Using Gaal-Koksma's method (see Theorem 1.1.3 in Section 1.1.3) yields

$$\sum_{i \le n} (X_i - \mathbf{E} X_i) = \mathrm{O}(n^{1/2} (\log n)^{3/2+\epsilon})$$

for all $\epsilon > 0$ and together with (3.17) we get the result. □

Proof of Corollary 3.2.1. First, we can assume w.l.o.g. that $\deg S < \deg R$. Furthermore, choose a polynomial T with $\deg T < \deg R$ and $(S, T, R) = 1$. Define a function as

$$f(C_1, C_2, A) = \begin{cases} 1 & \text{if } C_1 = S \text{ and } C_1 = T \\ 0 & \text{otherwise} \end{cases}$$

for polynomials C_1, C_2 with $\deg C_1, \deg C_2 < \deg R$ and extend it mod R in the first two coordinates. By applying Theorem 3.2.3 the Corollary follows. □

3.2.4 LeVeque's Extended Problem for Formal Laurent Series

We follow the proof in Section 2.1, whereas we mainly concentrate on differences. The first steps are very similar to the ones performed in Section 3.1.3 and we just replace the sequence $(U_k)_{k \ge 0}$ in Section 3.1.3 by

$$U_k(\alpha) := \#\{0 \le \deg C | CQ_k \equiv S \ (R), |C|^2 < |A_{k+1}| f(\deg CQ_k)\}$$

and the sequence $(V_k^{(i)})_{k \ge 0}$ by

$$V_k^{(i)}(\alpha) := \#\{0 \le \deg C | CQ_k \equiv S \ (R), |C|^2 < |A_{k+1}| f_i(k)\}$$

The next step in Section 2.1.2 was to approximate ones more by a double sequence of random variables and we are going to follow this method here. Therefore, we put

$$F_i(n) = \sum_{k=0}^{n} q^{\lceil \log_q f_i(k) \rceil}$$

and define

$$V_{k,n}^{(i)}(\alpha) := \#\{0 \leq \deg C \leq \phi_n | CQ_k \equiv S\ (R), |C|^2 < |A_{k+1}| f_i(k)\} \tag{3.21}$$

where $\phi_n = [\log_q((F_i(n))^{1/2}(\log F_i(n))^{1/2-\rho})]$ and $\rho > 0$ is a real constant.

In Section 2.1.2, moments of the above sequence of random variables were computed with help of several lemmas. We need analogues of these lemmas. The first lemma was the Theorem of Szüsz and we will use Theorem 3.2.1 instead of it. The second lemma (compare with Lemma 2.1.1 of Section 2.1.2) was an identity observed by Philipp. We have the following analogue in the Laurent series case

Theorem 3.2.7. *Set*

$$K(D) := \frac{|R|\varphi((D,R))}{C(R)|(D,R)|}.$$

Then, we have

$$\sum_{\deg D<\deg R} \sum_{\substack{0 \leq \deg C \\ CD \equiv S\ (R)}} \frac{K(D)}{|C|^2} = \frac{q}{|R|}. \tag{3.22}$$

Remark 3.2.4. Philipp obtained this identity in the classical case by comparing two deep results on metric diophantine approximation. We give here a direct and elementary proof.

For the proof, we need a few lemmas

Lemma 3.2.4. *Assume that* $\deg S < \deg R$ *and set*

$$L = \{C \in \mathbb{F}_q[T] | C \equiv S\ (R)\}.$$

Then, we have

$$\sum_{C\in L} \frac{1}{|C|^2} = \frac{1}{|S|^2} + \frac{q}{|R|^2} \tag{3.23}$$

if $\deg S \geq 0$ *and*

$$\sum_{C\in L, C\neq 0} \frac{1}{|C|^2} = \frac{q}{|R|^2} \tag{3.24}$$

if $S = 0$*, respectively.*

Proof. This is an easy calculation. □

Lemma 3.2.5. *We have*

$$\sum_{(C,R)=1, 0\leq \deg C<\deg R} \frac{1}{|C|^2} = \frac{q}{|R|^2}(C(R) - \varphi(R)). \tag{3.25}$$

Proof. We use the principle of inclusion and exclusion in order to get

$$\sum_{(C,R)=1,0\leq \deg C<\deg R} \frac{1}{|C|^2} = \sum_{P_1\cdots P_k|R} (-1)^k \sum_{0\leq C<\deg R-\deg P_1\cdots P_k} \frac{1}{|P_1\cdots P_k C|^2}$$

where the first sum on the right hand side is extended over all monic, irreducible, pairwise disjoint k-tuples of polynomials $P_1,\ldots,P_k$ satisfying $P_1\cdots P_k|R$. A straightforward calculation gives

$$\sum_{0\leq C<\deg R-\deg P_1\cdots P_k} \frac{1}{|P_1\cdots P_k C|^2} = q\left(\frac{1}{|P_1\cdots P_k|^2} - \frac{1}{|R||P_1\cdots P_k|}\right)$$

and consequently

$$\begin{aligned}
\sum_{(C,R)=1,0\leq \deg C<\deg R} \frac{1}{|C|^2} &= q\sum_{P_1\cdots P_k|R} \frac{(-1)^k}{|P_1\cdots P_k|^2} - \frac{q}{|R|}\sum_{P_1\ldots P_k|R}\frac{(-1)^k}{|P_1\ldots P_k|} \\
&= q\prod_{P|R}\left(1-\frac{1}{|P|^2}\right) - \frac{q}{|R|}\prod_{P|R}\left(1-\frac{1}{|P|}\right) \\
&= \frac{q}{|R|^2}(C(R)-\varphi(R))
\end{aligned}$$

which proves the result. □

Lemma 3.2.6. *We have*

$$\sum_{\deg C<\deg R,(C,R)|S} |(C,R)|\varphi((C,R)) = |(S,R)|\varphi(R)\prod_{P|R,P\nmid\frac{R}{(S,R)}}\left(1+\frac{1}{|P|}\right) \tag{3.26}$$

where the product is extended over all monic, irreducible polynomials P satisfying the desired property.

Proof. The proof of Lemma 2.1.3 of Section 2.1.2 carries over without difficulties. □

After these preliminaries, we can start with the proof of Theorem 3.2.7.

Proof of Theorem 3.2.7. First, observe that we can add the condition $(D,R)|(S,R)$ to the first sum on the left hand side of (3.22) because otherwise the congruence in the second sum has no solutions and therefore, the sum is 0 anyway. So, we have

$$\sum_{\deg D<\deg R}\ \sum_{\substack{0\leq \deg C\\ CD\equiv S\ (R)}} \frac{K(D)}{|C|^2} = \sum_{K|(S,R)}\ \sum_{\deg D<\deg R,(D,R)=K}\ \sum_{\substack{0\leq \deg C\\ C\bar{D}\equiv \bar{S}\ (\bar{R})}} \frac{K(D)}{|C|^2} \tag{3.27}$$

where $\bar{D} = D/K, \bar{S} = S/K, \bar{R} = R/K$ and the first sum on the right hand side runs over all monic divisors of (S, R). Next, we have to differ between $S = 0$ and $S \neq 0$.

In case $S = 0$, we compute

$$\begin{aligned}\sum_{K|R}\left(\sum_{\deg D<\deg R,(D,R)=K}\left(\sum_{C\equiv 0\ (\bar{R}),0\leq \deg C}\frac{K(D)}{|C|^2}\right)\right)\\ &= \frac{q}{|R|C(R)}\sum_{K|R}|K|\varphi(K)\varphi\left(\frac{R}{K}\right)\\ &= \frac{q}{|R|C(R)}\sum_{\deg C<\deg R}|(C,R)|\varphi((C,R)) = \frac{q}{|R|}\end{aligned}$$

where Lemma 3.2.4 and Lemma 3.2.6 were used.

In case $S \neq 0$, notice that the last sum on the right hand side of (3.27) runs over a residue class $\bar{E}$ mod $\bar{R}$ with $(\bar{E}, \bar{R}) = (\bar{S}, \bar{R})$. Conversely, it is easy to see that each residue class $\bar{E}$ mod $\bar{R}$ with $(\bar{E}, \bar{R}) = (\bar{S}, \bar{R})$ appears on the right hand side of (3.27) and moreover, each residue class with the above equally often property appears and therefore

$$T(K) := \varphi\left(\frac{R}{K}\right)/\varphi\left(\frac{R}{(S,R)}\right)$$

many times. Hence

$$\begin{aligned}\sum_{K|(S,R)}\sum_{\substack{\deg D<\deg R\\(D,R)=K}}\sum_{\substack{0\leq \deg C\\ C\bar{D}\equiv \bar{S}\ (\bar{R})}}\frac{K(D)}{|C|^2}\\ = \frac{|R|}{C(R)}\sum_{K|(S,R)}\frac{\varphi(K)T(K)}{|K|}\sum_{\substack{\deg \bar{E}<\deg \bar{R}\\(\bar{E},\bar{R})=(\bar{S},\bar{R})}}\sum_{C\equiv \bar{E}\ (\bar{R})}\frac{1}{|C|^2}.\end{aligned}$$

Because of Lemma 3.2.4 and Lemma 3.2.5, we have for the last two sums

$$\begin{aligned}\sum_{\substack{\deg \bar{E}<\deg \bar{R}\\(\bar{E},\bar{R})=(\bar{S},\bar{R})}}\sum_{C\equiv \bar{E}\ (\bar{R})}\frac{1}{|C|^2} &= \sum_{\substack{\deg \bar{E}<\deg \bar{R}\\(\bar{E},\bar{R})=(\bar{S},\bar{R})}}\left(\frac{1}{|\bar{E}|^2}+\frac{q}{|\bar{R}|^2}\right)\\ &= \frac{\varphi(\tilde{R})q}{|\bar{R}|^2}+\sum_{\substack{\deg E<\deg \tilde{R}\\(E,\tilde{R})=1}}\frac{1}{|(\bar{S},\bar{R})E|^2}\\ &= \frac{\varphi(\tilde{R})q}{|\bar{R}|^2}+\frac{q}{|(\bar{S},\bar{R})\tilde{R}|^2}(C(\tilde{R})-\varphi(\tilde{R}))\\ &= \frac{q}{|\bar{R}|^2}C(\tilde{R})\end{aligned}$$

where $\tilde{R} = R/(S,R)$. Consequently, by using Lemma 3.2.6

$$\sum_{K|(S,R)} \sum_{\substack{\deg D < \deg R \\ (D,R)=K}} \sum_{\substack{0 \le \deg C \\ C\bar{D} \equiv \bar{S}\ (\bar{R})}} \frac{K(D)}{|C|^2}$$

$$= \frac{qC(\tilde{R})}{|R|C(R)\varphi(\tilde{R})} \sum_{K|(S,R)} |K|\varphi(K)\varphi(\bar{R})$$

$$= \frac{qC(\tilde{R})}{|R|C(R)\varphi(\tilde{R})} |(S,R)|\varphi(R) \prod_{P|R, P\nmid \tilde{R}} \left(1 + \frac{1}{|P|}\right)$$

and after a little bit straightforward calculation we are done. □

Since Lemma 3.2.6 is already an analogue for Lemma 2.1.3 of Section 2.1.2 and we don't need an analogue for Lemma 2.1.2 of Section 2.1.2 in order to compute moments of (3.21), we are ready to prove the following result (compare with Lemma 2.1.4 in Section 2.1.2)

Lemma 3.2.7. *For the sequence (3.21) we have*

$$\mu_{i,n} := \mathbf{E} \sum_{k \le n} V_{k,n}^{(i)} = \frac{q}{|R|} F_i(n) + O\left(F_i(n)^{1/2}\right),$$

and

$$\tau_{i,n}^2 := \mathbf{V} \sum_{k \le n} V_{k,n}^{(i)} \sim \sigma F_i(n) \log F_i(n), \tag{3.28}$$

with

$$\sigma = \frac{(q^2-1)|(S,R)|\varphi(R)}{2 \log q |R| C(S,R)}.$$

Proof. The proof is similar to the one of Lemma 2.1.4 in Section 2.1.2 and so, we are only going to point out differences, whereas we concentrate on (3.28). Therefore, write

$$V_{k,n}^{(i)} = \sum_{\substack{\deg D < \deg R \\ (D,R)|S}} \sum_{\substack{0 \le \deg C \le \phi_n \\ CD \equiv S\ (R)}} \xi_k^{(C,D,i)}$$

where

$$\xi_k^{(C,D,i)}(\alpha) = \begin{cases} 1 & \text{if } Q_k \equiv D\ (R) \text{ and } |C|^2 < |A_{k+1}| f_i(k), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\mathbf{E}(V_{k,n}^{(i)})^2 = \sum_{\substack{\deg D < \deg R \\ (D,R)|S}} \sum_{l=\deg \bar{R}}^{\phi_n} k_{l,D} K(D) q^{\lceil \log_q f_i(k) \rceil - 2l} (1 + \mathrm{O}(\rho^k))$$

$$+ \mathrm{O}\left(q^{\lceil \log_q f_i(k) \rceil}\right).$$

Here $\bar{R} = R/(D, R)$ and

$$k_{l,D} = (q^2 - 1)^2 q^{2(l - \deg \bar{R})} + \mathrm{O}(q^{l - \deg \bar{R}})$$

where the implied constant is absolute. Consequently, after a little bit straightforward calculation,

$$\begin{aligned}\mathbf{E}(V_{k,n}^{(i)})^2 &= q^{\lceil \log_q f_i(k) \rceil}(1 + \mathrm{O}(\rho^k)) \frac{(q^2-1)\phi(n)}{|R|C(R)} \\ &\qquad \sum_{\substack{\deg D < \deg R \\ (D,R)|S}} |(D,R)|\varphi((D,R)) + \mathrm{O}\left(q^{\lceil \log_q f_i(k) \rceil}\right) \\ &= q^{\lceil \log_q f_i(k) \rceil}(1 + \mathrm{O}(\rho^k)) \frac{(q^2-1)\phi(n)|(S,R)|\varphi(R)}{|R|C(S,R)} \\ &\qquad + \mathrm{O}\left(q^{\lceil \log_q f_i(k) \rceil}\right)\end{aligned}$$

where Lemma 3.2.6 was used. Next, we sum over k and hence

$$\sum_{k \leq n} \mathbf{E}(V_{k,n}^{(i)})^2 = \frac{(q^2-1)\phi(n)|(S,R)|\varphi(R)}{|R|C(S,R)} F_i(n) + \mathrm{O}(\phi_n) + \mathrm{O}(F_i(n)).$$

Therefore, we have

$$\sum_{k \leq n} \mathbf{E}(V_{k,n}^{(i)})^2 \sim \sigma F_i(n) \log F_i(n)$$

and we can proceed as in the proof of Lemma 2.1.4 of Section 2.1.2 in order to get (3.28). □

In Section 2.1.2, the analogue of this result together with the mixing behaviour of the sequence analogues to (3.21) was used to obtain Lemma 2.1.5. Lemma 2.1.5 in turn was used to prove the central limit theorem for the approximating sequence of random variables analogues to (3.21). Since (3.21) has exactly the same mixing behaviour (follows immediatly from Theorem 3.2.2), the proof method introduced in Section 2.1.3 can be used without difficulties to get asymptotic normality of (3.21).

Lemma 3.2.8. *We have*

$$\lim_{n \longrightarrow \infty} h \left[\left(\sum_{k \leq n} V_{k,n}^{(i)} - \mu_{i,n} \right) / \tau_{i,n} \leq \omega \right] = \Phi(\omega).$$

Furthermore, Lemma 2.1.6 of Section 2.1.4 remains true in our situation with a similar proof and therefore, we can conclude the proof of Theorem 3.2.4 as in Section 2.1.4 by using the Lemma 2.1.7 of Section 3.1.3.

Lemma 3.2.9. *Let g_1 (resp. g_2) be the inverse function of $(k+1)\log\gamma + \kappa(k+1)^{1-\delta}$ (resp. $k\log\gamma - \kappa k^{1-\delta}$). Then we have*

$$F_i(g_i(n)) = \frac{1}{\log\gamma}F(n) + O(F(n)^{1/2}).$$

3.2.5 Invariance Principles in the Laurent Series Case

In order to prove Theorem 3.2.5 and Theorem 3.2.6, we follow the proofs of the corresponding theorems in Section 2.2, whereas we again only emphasize differences. First define

$$V_k^{(i)}(\alpha) := \#\{0 \le \deg C \le d | CQ_k \equiv S\ (R), |C|^2 \le |A_{k+1}| f_i(k)\} \tag{3.29}$$

where $f_i(k)$ is defined as in the last section (resp. in Section 3.1.3). Similarly to the first steps performed in Section 3.1.3 one can use the above sequences of random variables to approximate $X_n^{(d)}$ by a suitable sum of $V_k^{(i)}$.

In Section 2.2, we proceeded in two steps. In a first step, we proved almost sure and distribution type invariance principles for the approximating sequence. In a second step, we proved the corresponding results for the sequence analogues to $X_n^{(d)}$ by using the approximation. Thereby, the second step was only technical and the method can be used without difficulties in the Laurent series case, too. Though the method of the first step could also be carried over in the Laurent series case, we give different proofs because here the situation is much more easier. This is thanks to the fact that the approximating sequence (3.29) is $\star$-mixing with an exponential mixing rate in difference to the approximating sequence in Section 2.2 that did not possess this property.

So, the following analogue of Lemma 2.2.4 in Section 2.2.2 can be obtained just by using Theorem 3.2.1 and some straightforward calculation (in difference to the proof of the result in Section 2.2.2 where we had to approximate once more)

Lemma 3.2.10. *Set*

$$F_i(n) = \sum_{k=0}^{n} q^{\lceil \log_q f_i(k) \rceil}.$$

Then, we have

$$\mathbf{E}\sum_{k=0}^{n} V_k^{(i)} = (\sigma\log\gamma)F_i(n) + O(1), \tag{3.30}$$

and either

$$\mathbf{V}\sum_{k=0}^{n} V_k^{(i)} \sim (\tau^2\log\gamma)F_i(n), \tag{3.31}$$

if f satisfies (1.25) and (1.26), or

$$\mathbf{V}\sum_{k=0}^{n} V_k^{(i)} = (\tau^2 \log\gamma)F_i(n) + O(F_i(n)^{1-\delta_3}),$$

if f satisfies (1.25), (1.26), and (2.39).

Proof. Similar to the proof of Lemma 2.2.4 in the last section. □

Next, we normalize

$$\xi_k^{(i)} = V_k^{(i)} - \mathbf{E}V_k^{(i)}$$

and define a suitable blocking (compare with the definition of η_k in Section 2.2.3): for a fixed positive integer k denote by h_k the integer satisfying

$$(\tau^2 \log\gamma)F_i(h_k) \leq k < (\tau^2 \log\gamma)F_i(h_k + 1)$$

and set $h_0 = -1$. Then, define a sequence of random variables as

$$\eta_k^{(i)} = \sum_{l=h_{k-1}+1}^{h_k} \xi_l^{(i)}.$$

Using this notation, we can prove the following analogue of Lemma 2.2.5 in Section 2.2.3

Lemma 3.2.11. *We have*

1. *The sequence $(\eta_k^{(i)})_{k\geq 1}$ is $\star$-mixing with an exponential mixing rate.*
2. *Either*

$$\mathbf{E}\left(\sum_{k=1}^{n}\eta_k^{(i)}\right)^2 \sim n,$$

if f satisfies (1.25) and (1.26), or

$$\mathbf{E}\left(\sum_{k=1}^{n}\eta_k^{(i)}\right)^2 = n + O(n^{1-\delta_3}),$$

if f satisfies (1.25), (1.26), and (2.39).

3. $$\mathbf{E}\left|\eta_k^{(i)}\right|^3 \ll 1.$$

Proof. (1) is obvious since $\xi_l^{(i)}$ is $\star$-mixing with an exponential mixing rate (follows immediatly from Theorem 3.2.2) and (2) is a consequence of the last lemma together with the definition of $\eta_k^{(i)}$.

In order to prove (3) notice

$$0 \leq k - (\tau^2 \log \gamma) F_i(h_k) \leq (\tau^2 \log \gamma) f_i(h_k + 1) \ll 1$$

and therefore

$$(\tau^2 \log \gamma) F_i(h_k) = k + \mathrm{O}(1). \tag{3.32}$$

Next, observe by the multinomial theorem

$$\begin{aligned}
\mathbf{E}\left|\eta_k^{(i)}\right|^3 &= \mathbf{E}\left|\sum_{l=h_{k-1}+1}^{h_k} \xi_l^{(i)}\right|^3 \\
&= \sum_{e_{h_{k-1}+1}+\ldots+e_{h_k}=3} \binom{3}{e_{h_{k-1}+1}, \ldots, e_{h_k}} \mathbf{E}\left|\xi_{h_{k-1}+1}^{(i)}\right|^{e_{h_{k-1}+1}} \cdots \left|\xi_{h_k}^{(i)}\right|^{e_{h_k}} \\
&\ll \sum_{e_{h_{k-1}+1}+\ldots+e_{h_k}=3} \binom{3}{e_{h_{k-1}+1}, \ldots, e_{h_k}} \mathbf{E}\left|\xi_{h_{k-1}+1}^{(i)}\right|^{e_{h_{k-1}+1}} \cdots \mathbf{E}\left|\xi_{h_k}^{(i)}\right|^{e_{h_k}} \\
&\ll \left(\sum_{l=h_{k-1}+1}^{h_k} \mathbf{E}\left|\xi_l^{(i)}\right|\right)^3 \ll (F_i(h_k) - F_i(h_{k-1}))^3 \ll 1
\end{aligned}$$

which is the desired result. Here the mixing property of $\xi_k^{(i)}$, the following estimation which is straightforward to prove

$$\mathbf{E}\left|\xi_k^{(i)}\right| \ll \mathbf{E} V_k^{(i)} \ll q^{\lceil \log_q f_i(k) \rceil},$$

and (3.32) were used. □

For the rest of the proof of almost sure and distribution type invariance principles for the sequence (3.29), we can proceed as in Section 2.2.3 and therefore, with the remarks at the beginning of this section, we are done.

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Acknowledgment

Here, I want to express my sincere appreciation to those who are in one way or the other related to this work. First of all, I am very much indebted to Prof. Michael Drmota for supervising this thesis and providing continuosly support. Furthermore, I cordially would like to thank Prof. Walter Philipp for pointing my attention to the results in [24] on which now all theorems in Section 2.2 are based and for valuable suggestions and fruitful discussions about Section 2.2. Finally, I am grateful to the Austrian Science Foundation FWF, grant S8302-MAT for giving financial support.

Last but surely not least I am very thankful to all my friends and my family for their permanently encouragement and guidance; especially, to Sun Hee who - beside many other things I am thankful for - corrected misprints in the first draft of this work and brought my english in a more native form.

Vienna, February 2002 Michael Fuchs

Lebenslauf (German)

Ich wurde am 26. September 1975 als Sohn von Notburga Fuchs, geb. Salcher, und Josef Fuchs in Lienz/Tirol geboren. Von 1982 bis 1986 besuchte ich die Volksschule in Assling, in den anschließenden 4 Jahren das Bundesrealgymnasium in Lienz und zwischen 1990 und 1995 die HTBLVA in Innsbruck, Fachrichtung Nachrichtentechnik und Leistungselektronik, die ich am 14. Juni 1995 durch Ablegung der Matura mit Auszeichnung beendete.

Im Oktober 1995 immatrikulierte ich an der Technischen Universität Wien und inskribierte für das Studium Technische Mathematik, Studienzweig Mathematische Computerwissenschaften. Am 9. Mai 2000 schloss ich dieses Diplomstudium mit Auszeichnung ab.

In den Jahren 1999 bis 2002 war ich am Institut für Angewandte und Numerische Mathematik der Technischen Universität Wien als Studienassistent tätig. Seit Mai 2000 arbeite ich als Forschungsassistent in dem FWF-Projekt *Statistical Properties of Number Systems* unter der Leitung von Prof. Michael Drmota.

Wien, Februar 2002 Michael Fuchs