Combinatorial comparison of general galled trees, time-consistent galled trees, and simplex time-consistent galled trees

Lily Agranat-Tamir*
Michael Fuchs†
Bernhard Gittenberger‡
Noah A. Rosenberg*
Karthik V. Seetharaman*

December 16, 2025

Abstract. Rooted binary phylogenetic networks are extensions of rooted binary trees, adding reticulation nodes that are designed to represent evolutionary processes that involve hybridization events. Enumerative combinatorics studies have counted leaf-labeled phylogenetic networks in a variety of classes, finding that when the number of reticulations is fixed, the time-consistent galled trees are asymptotically less numerous than each of several network classes that had been previously examined. Here we provide enumerative results on two additional network classes: general galled trees and simplex time-consistent galled trees. We show that for a fixed number of galls, as the number of leaves goes to infinity, the asymptotic count of general galled trees is identical to that of time-consistent galled trees, whereas the count of simplex time-consistent galled trees is smaller. If the number of galls is not restricted, then the asymptotic approximations all differ: simplex time-consistent galled trees are less numerous than time-consistent galled trees, which are in turn less numerous than general galled trees. We also report a variety of additional results: recursions to count the studied networks with small numbers of leaves a fixed number of galls, as well as enumerative results for unlabeled networks in the classes that we investigate.

Key words: galled trees, leaf-labeled trees, phylogenetic networks, symbolic method, unlabeled trees

Mathematics subject classification (2020): 05A16, 05A15, 05C05, 92D15

*Department of Biology, Stanford University, Stanford, CA 94305 USA
†Department of Mathematical Sciences, National Chengchi University, Taipei, Taiwan
‡Department of Discrete Mathematics and Geometry, Technische Universität Wien, Austria

1 Introduction

Rooted binary phylogenetic networks are rooted binary trees with the addition of nodes of in-degree 2 and out-degree 1, *reticulation* or *hybrid nodes*, which produce *reticulation cycles*. These networks are graph structures that are used to study evolutionary processes in which biological lineages that have previously diverged come back together (reviewed by [22]).

Many classes of phylogenetic networks have had their enumerative properties studied mathematically [1, 2, 3, 4, 5, 6, 7, 8, 12, 13, 14, 15, 17, 18, 19, 20, 23, 24, 29]. In recent work, we have focused on one such class, the *time-consistent galled trees* [1, 2, 3, 24]. In *galled trees*, each reticulation cycle shares no vertices or edges with other such cycles, and is known as a *gall* [1, 2, 24]. In *time-consistent* galled trees, the two nodes that are immediately ancestral to a reticulation node—its *hybridizing nodes*—do not have an ancestor–descendant relation, so that they can be viewed as contemporaneous if a sense of time is imposed on the network. *Time-consistent* galled trees are equivalent to *normal* galled trees.

A hallmark property in enumerative combinatorics, namely the asymptotic approximation of the size of a combinatorial class as a suitable variable grows in size, has been studied in several classes of phylogenetic networks: reticulation-visible networks, tree-child networks, normal networks, galled networks, galled tree-child networks, and time-consistent galled trees, as well as in the class of general phylogenetic networks. Specifically, the numbers of leaf-labeled general phylogenetic networks [23], reticulation-visible networks [7], tree-child networks [15], normal networks [15, 17], galled networks [7], and galled tree-child networks [8], all with a fixed number k of reticulation nodes, as the number of leaves n goes to infinity, have the same asymptotic growth. Denoting the counts for these various classes by PN, RV, TC, N, GN, and GTC, respectively, as $n \to \infty$, the enumerations satisfy:

$$PN_{n,k} \sim RV_{n,k} \sim TC_{n,k} \sim N_{n,k} \sim GN_{n,k} \sim GTC_{n,k} \sim \frac{2^{k-1}\sqrt{2}}{k!} n^{n+2k-1} \left(\frac{2}{e}\right)^n. \tag{1}$$

We found, however, that the corresponding approximation for the number of leaf-labeled time-consistent galled trees with a fixed number of galls—equivalent to the number of reticulation nodes—differs. In particular, the subexponential part of the asymptotic expansion is smaller by a factor of $(2k)!/(2^k k!)$ [2, eq. (53)]. Denoting the count of time-consistent galled trees by C,

$$C_{n,k} \sim \frac{2^{2k-1}}{(2k)!\,\sqrt{\pi}} n^{2k-\frac{3}{2}} \left(\frac{1}{2}\right)^{-n} n! \sim \frac{2^{2k-1}\sqrt{2}}{(2k)!} n^{n+2k-1} \left(\frac{2}{e}\right)^n. \tag{2}$$

In this study, we consider the enumerative properties for two additional classes of galled trees: one that contains the time-consistent galled trees as a proper subset, and one that is itself a proper subset of the time-consistent galled trees. For the galled trees in general, or *general galled trees*—which contain the time-consistent galled trees—does the asymptotic enumeration follow the larger eq. (1), like larger classes of phylogenetic networks, or the smaller eq. (2), like the time-consistent galled trees? For the class known as the *simplex time-consistent galled trees*, which are contained among the time-consistent galled trees, does the asymptotic enumeration follow the time-consistent galled trees as in eq. (2), or is it still smaller?

In Section 3, we count unlabeled general galled trees, and in Section 4, we count leaf-labeled general galled trees. In Section 5, we count unlabeled simplex time-consistent galled trees, and in Section 6, we count leaf-labeled simplex time-consistent galled trees. In particular, we analyze the asymptotic approximations of the counts of these networks as the number of leaves goes to infinity, either with a fixed number of galls (Sections 3.5 and 4.5 for general galled trees and Sections 5.5 and 6.5 for simplex time-consistent galled trees)

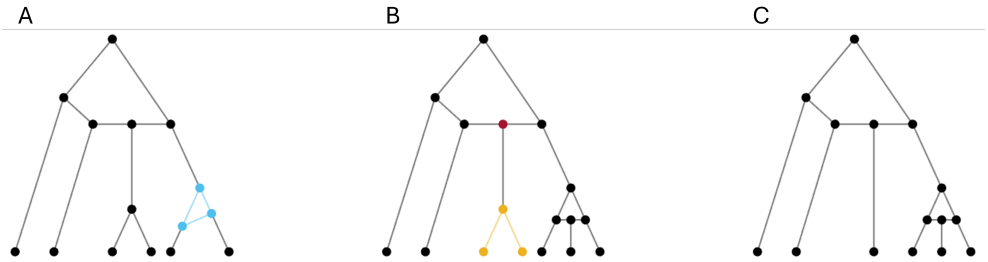

Figure 1: Three classes of phylogenetic networks. (A) A general galled tree. This galled tree is not a time-consistent galled tree because it has a gall (light blue) whose top node is not separated from the reticulation node by at least two edges (it is a parent of the reticulation node). (B) A time-consistent galled tree. This tree is not a simplex galled tree because it has a reticulation node (red) that is the ancestor of a subtree that is not a single leaf (orange). (C) A simplex time-consistent galled tree.

or with any number of galls (Sections 3.6 and 4.6 for general galled trees and Sections 5.6 and 6.6 for simplex time-consistent galled trees). Section 4.6, covering all numbers of galls in leaf-labeled general galled trees, repeats analysis conducted by [5] and [12]; the latter study in addition contains results for the unlabeled case of Section 3.6. Finally, in Section 7, we find recursions that we use to obtain the exact numbers of the various networks for small numbers of leaves.

The analysis confirms that general galled trees with a fixed number of galls behave asymptotically in the same way as time-consistent galled trees with a fixed number of galls. However, simplex time-consistent galled trees with a fixed number of galls behave differently. The three classes of networks differ asymptotically when no restrictions are imposed on the number of galls, that is, when looking at the union of networks with fixed numbers of galls.

2 Definitions

The study builds from [2] and uses the same terms, where appropriate. Many definitions for phylogenetic networks were provided by [22]. We consider "non-plane" networks, in which changing the left–right order in which nodes are graphically depicted does not change the network.

A *rooted binary phylogenetic network* is a simple directed rooted graph with the following four types of nodes: (1) a unique *root* node with in-degree 0 and out-degree 2; (2) *tree* nodes with in-degree 1 and out-degree 2; (3) *leaf* nodes with in-degree 1 and out-degree 0; (4) *reticulation* nodes with in-degree 2 and out-degree 1. As a simple graph, a phylogenetic network has no edges that connect a node to itself, and each pair of nodes connected by an edge are connected by precisely one edge. A rooted binary tree possesses the first three types of nodes; a phylogenetic network can also have the fourth type. We sometimes refer to the rooted binary phylogenetic networks as *general phylogenetic networks*.

The direction of the network is from the root to the leaves. We sometimes refer to reticulation nodes as *hybrid* nodes; the two parents of a hybrid node are called *hybridizing* nodes. A *reticulation cycle* is a pair of directed paths with common start and end nodes that are otherwise node-disjoint.

Galled trees are rooted binary phylogenetic networks with the additional property that each node is found in at most one reticulation cycle (Figure 1A). In galled trees, reticulation cycles are called *galls*. A gall possesses an ancestor node (the "top" node), from which the two directed paths in the gall originate, and a hybrid node where these paths intersect.

Technically, galled trees that possess reticulation nodes are not actually trees in the mathematical sense; in accord with the term "galled trees," we refer to the galled trees rooted at internal nodes of a galled tree as *subtrees*; these subtrees also need not be trees in a technical sense. For clarity, we sometimes refer to galled trees as *general galled trees*.

Time-consistent galled trees are galled trees with the additional property that the ancestor node r of a gall, from which the two disjoint paths reach the reticulation node a_r, is separated from a_r by two or more edges (Figure 1B). This last condition encodes the requirement that we consider only *normal* galled trees; *normal* galled trees are equivalent to *time-consistent* galled trees. The definition of time-consistent galled trees follows d [1], [3], and [24].

Simplex galled trees, also known as *simplicial galled trees* or *1-component galled trees*, are galled trees for which the subtree of each reticulation node is a single leaf (Figure 1C; [12, 16, 30]). Henceforth we refer to this class of trees with the term *simplex* galled trees. Simplex galled trees are suited to scenarios with recent hybridization events, after which insufficient time has passed for lineages descended from reticulation nodes to give rise to further reticulations.

For time-consistent galled trees, viewing each gall as a representation of a biological merging event, for each gall we depict its hybridizing nodes and hybrid node on a horizontal line. This depiction represents the simultaneity of the hybridizing and hybrid nodes when a galled tree is viewed as a structure evolving in time. Such a depiction is possible in time-consistent networks but not in galled trees for which the two parents of a reticulation node have a parent–child relation (Figure 1A).

In the network classes that we study—general galled trees (G), time-consistent galled trees (C), and simplex time-consistent galled trees (SC)—we consider both unlabeled and leaf-labeled networks. We stated the definition of general phylogenetic networks (PN) above; although we do not need formal definitions of reticulation-visible networks (RV), tree-child networks (TC), normal networks (N), galled networks (GN), and galled tree-child networks (GTC), it is useful to specify inclusion relationships: the set of galled trees G is a subset of the classes GTC, GN, TC, RV, and PN. The set of time-consistent galled trees C is a subset of G and of N. The set of simplex time-consistent galled trees SC is a subset of C.

3 Unlabeled general galled trees

Note that whereas the number of galls g for time-consistent galled trees n satisfies $0 \leq g \leq \lfloor \frac{n-1}{2} \rfloor$ (Section 2.4 of [3]), for general galled trees, the number of galls satisfies $0 \leq g \leq n-1$. This result can be obtained by noting that the smallest general galled tree has a root gall in which two nodes descend from the root, one of which is a reticulation node, and from both of which a single leaf descends ($n = 2$, $g = 1$). Each subsequent gall replaces a leaf with a minimum of two leaves; a galled tree with n leaves and $n-1$ galls can be constructed by sequentially replacing leaves with triangular galls from which two leaves descend, so that a general galled tree with n leaves has at most $n-1$ galls.

3.1 No galls

The unlabeled general galled trees with no galls are the unlabeled binary trees. The generating function $\mathcal{U}(t) = \sum_{n=0}^{\infty} U_n t^n$, whose term U_n counts unlabeled binary trees with n leaves, satisfies [21, 27, p. 55],

$$\mathcal{U}(t) = t + \frac{1}{2}[\mathcal{U}(t^2) + \mathcal{U}(t)^2].$$

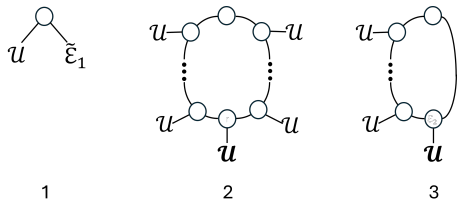

Figure 2: The three structures possible for a (non-plane) unlabeled general galled tree with exactly one gall.

Asymptotically, as $t \to \rho^-$, the generating function has equivalence

$$\mathcal{U}(t) \sim 1 - \gamma\Big(1 - \frac{t}{\rho}\Big)^{\frac{1}{2}}, \tag{3}$$

where $\rho \approx 0.40270$ and $\gamma \approx 1.13003$ [11, pp. 476-477]. Further, as $n \to \infty$,

$$U_n = [t^n]\mathcal{U}(t) \sim \frac{\gamma}{2\sqrt{\pi}} n^{-\frac{3}{2}} \rho^{-n}. \tag{4}$$

The U_n are the Wedderburn–Etherington numbers (OEIS A001190). The first terms appear in Table 1.

3.2 One gall

In the case of one gall, if the root is not part of the gall, then as in the time-consistent case [2], the root has two children, one of which has one gall and the other of which is a binary tree (component 1 in eq. (5) and Figure 2). If the root is part of the gall, then two sequences of trees lead from the root to the reticulation node. Either both sequences of trees are nonempty, as in time-consistent galled trees (component 2), or one is nonempty and the other is empty (component 3).

We follow the symbolic method for enumerating unlabeled structures, as used by [2] for unlabeled time-consistent galled trees. In symbolic notation [11, p. 93],

$$\tilde{\mathcal{E}}_1 = \{\circ\} \times \big[\underbrace{\mathcal{U} \times \tilde{\mathcal{E}}_1}_{1} \;\dot\cup\; \underbrace{\mathcal{U} \times \mathrm{MSET}_2\big(\mathrm{SEQ}^+(\mathcal{U})\big)}_{2} \;\dot\cup\; \underbrace{\mathcal{U} \times \mathrm{SEQ}^+(\mathcal{U})}_{3}\big]. \tag{5}$$

Converting to a generating function $\tilde{\mathcal{E}}_1(t) = \sum_{n=0}^{\infty} \tilde{E}_{n,1} t^n$,

$$\tilde{\mathcal{E}}_1(t) = \mathcal{U}(t)\,\tilde{\mathcal{E}}_1(t) + \frac{\mathcal{U}(t)}{2}\left[\left(\frac{\mathcal{U}(t)}{1-\mathcal{U}(t)}\right)^2 + \frac{\mathcal{U}(t^2)}{1-\mathcal{U}(t^2)}\right] + \frac{\mathcal{U}(t)^2}{1-\mathcal{U}(t)}.$$

Solving for $\tilde{\mathcal{E}}_1(t)$, we obtain

$$\tilde{\mathcal{E}}_1(t) = \frac{\mathcal{U}(t)^3}{2[1-\mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)\,\mathcal{U}(t^2)}{2[1-\mathcal{U}(t)]\,[1-\mathcal{U}(t^2)]} + \frac{\mathcal{U}(t)^2}{[1-\mathcal{U}(t)]^2}. \tag{6}$$

In comparison with $\mathcal{E}_1(t) = \sum_{n=0}^{\infty} E_{n,1} t^n$, the generating function for the unlabeled time-consistent galled trees with one gall [2, eq. (6)], the generating function for the unlabeled general galled trees with one gall contains only the extra term $\mathcal{U}(t)^2/[1-\mathcal{U}(t)]^2$ from component 3 in eq. (5). The extra term is not a term of highest order; asymptotically, from eq. (3), as $t \to \rho^-$, $\tilde{\mathcal{E}}_1(t) \sim \mathcal{E}_1(t)$.

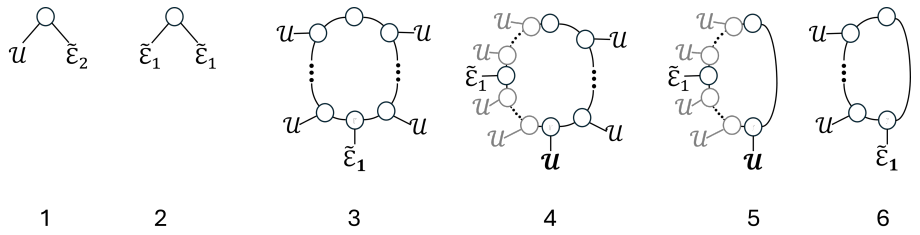

Figure 3: The six structures possible for a (non-plane) unlabeled general galled tree with exactly two galls.

From eq. (7) of [2], we conclude, as $n \to \infty$,

$$\tilde{E}_{n,1} = [t^n]\tilde{\mathcal{E}}_1(t) \sim \frac{1}{\gamma^3\sqrt{\pi}} n^{\frac{1}{2}} \rho^{-n}. \tag{7}$$

The first values of $\tilde{E}_{n,1}$ are calculated in Section 7.1. They appear in Table 1.

3.3 Two galls

For two galls, we have six cases (Figure 3). If the root node is not part of a gall, then its two children are either (1) a general galled tree with two galls and a general galled tree with zero galls, or (2) two general galled trees with one gall each. If the root node is part of a gall, then there are four cases: (3) both sequences from the root to the reticulation node are nonempty and the second gall is descended from the reticulation node; (4) both sequences from the root to the reticulation node are nonempty and the second gall is on one of these paths; (5) one sequence from the root to the reticulation node is nonempty and contains the second gall, and the other sequence is empty; (6) one sequence from the root to the reticulation node is nonempty, the other is empty, and the second gall is descended from the reticulation node. In symbolic notation,

$$\tilde{\mathcal{E}}_2 = \{\circ\} \times \big[\underbrace{\mathcal{U} \times \tilde{\mathcal{E}}_2}_{1} \;\dot\cup\; \underbrace{\mathrm{MSET}_2(\tilde{\mathcal{E}}_1)}_{2} \;\dot\cup\; \underbrace{\tilde{\mathcal{E}}_1 \times \mathrm{MSET}_2\big(\mathrm{SEQ}^+(\mathcal{U})\big)}_{3} \;\dot\cup\; \underbrace{\mathcal{U} \times \big(\mathrm{SEQ}(\mathcal{U}) \times \tilde{\mathcal{E}}_1 \times \mathrm{SEQ}(\mathcal{U})\big) \times \mathrm{SEQ}^+(\mathcal{U})}_{4}$$
$$\dot\cup\; \underbrace{\mathcal{U} \times \mathrm{SEQ}(\mathcal{U}) \times \tilde{\mathcal{E}}_1 \times \mathrm{SEQ}(\mathcal{U})}_{5} \;\dot\cup\; \underbrace{\tilde{\mathcal{E}}_1 \times \mathrm{SEQ}^+(\mathcal{U})}_{6}\big]. \tag{8}$$

Converting to a generating function $\tilde{\mathcal{E}}_2(t) = \sum_{n=0}^{\infty} \tilde{E}_{n,2} t^n$,

$$\tilde{\mathcal{E}}_2(t) = \mathcal{U}(t)\,\tilde{\mathcal{E}}_2(t) + \frac{1}{2}[\tilde{\mathcal{E}}_1(t)^2 + \tilde{\mathcal{E}}_1(t^2)] + \frac{\tilde{\mathcal{E}}_1(t)}{2}\left(\frac{\mathcal{U}(t)^2}{[1-\mathcal{U}(t)]^2} + \frac{\mathcal{U}(t^2)}{1-\mathcal{U}(t^2)}\right) + \frac{\mathcal{U}(t)^2\,\tilde{\mathcal{E}}_1(t)}{[1-\mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)\,\tilde{\mathcal{E}}_1(t)}{[1-\mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)\,\tilde{\mathcal{E}}_1(t)}{1-\mathcal{U}(t)}.$$

Solving for $\tilde{\mathcal{E}}_2(t)$, we get:

$$\tilde{\mathcal{E}}_2(t) = \frac{\tilde{\mathcal{E}}_1(t)^2}{2[1-\mathcal{U}(t)]} + \frac{\tilde{\mathcal{E}}_1(t^2)}{2[1-\mathcal{U}(t)]} + \frac{\mathcal{U}(t)^2\,\tilde{\mathcal{E}}_1(t)}{2[1-\mathcal{U}(t)]^3} + \frac{\mathcal{U}(t^2)\,\tilde{\mathcal{E}}_1(t)}{2[1-\mathcal{U}(t)]\,[1-\mathcal{U}(t^2)]} + \frac{\mathcal{U}(t)^2\,\tilde{\mathcal{E}}_1(t)}{[1-\mathcal{U}(t)]^4} + \frac{\mathcal{U}(t)\,\tilde{\mathcal{E}}_1(t)}{[1-\mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)\,\tilde{\mathcal{E}}_1(t)}{[1-\mathcal{U}(t)]^2}. \tag{9}$$

In comparison with $\mathcal{E}_2(t) = \sum_{n=0}^{\infty} E_{n,2} t^n$, the generating function for the unlabeled time-consistent galled trees with two galls [2, eq. (9)], the generating function for the unlabeled general galled trees with two galls contains two terms that arise from cases 5 and 6 in Figure 3. Neither is a term of highest degree; asymptotically, from eq. (3), as $t \to \rho^-$, $\tilde{\mathcal{E}}_2(t) \sim \mathcal{E}_2(t)$.

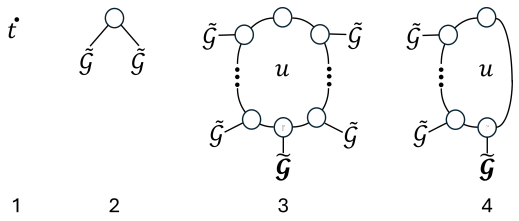

Figure 4: The four structures possible for a (non-plane) unlabeled general galled tree.

From eq. (10) of [2], we conclude, as $n \to \infty$,

$$\tilde{E}_{n,2} = [t^n]\tilde{\mathcal{E}}_2(t) \sim \frac{1}{3\gamma^7\sqrt{\pi}} n^{\frac{5}{2}} \rho^{-n}. \tag{10}$$

The first values of $\tilde{E}_{n,2}$ are calculated in Section 7.1, and they appear in Table 1.

3.4 Bivariate generating function

As can be deduced from the cases of one and two galls, the generating function for unlabeled general galled trees with any specific number of galls $g \geq 1$, $\tilde{\mathcal{E}}_g(t) = \sum_{n=0}^{\infty} \tilde{E}_{n,g} t^n$, is dependent on the corresponding generating functions for galled trees with fewer galls.

Therefore, to find $\tilde{\mathcal{E}}_g(t)$, we first find the bivariate generating function, $\tilde{\mathcal{G}}(t,u) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \tilde{E}_{n,m} t^n u^m$, counting both the number of leaves and the number of galls. We obtain the generating function $\tilde{E}_g(t)$ for a fixed number of galls g by noting $\mathcal{U}(t) = \tilde{\mathcal{E}}_0(t) = \tilde{\mathcal{G}}(t,0)$, and for $g \geq 1$,

$$\tilde{\mathcal{E}}_g(t) = \frac{1}{g!}\left(\frac{\partial^g}{\partial u^g}\tilde{\mathcal{G}}\right)(t,0).$$

In the symbolic method, general galled trees include the same components as time-consistent galled trees [2, eq. (14)], with the addition of a case that has a root gall and an empty sequence on one side of the reticulation node (Figure 4). The symbolic method gives us

$$\tilde{\mathcal{G}} = \underbrace{\{\square\}}_{1} \,\dot\cup\, \{\circ\} \times \Bigg[\underbrace{\mathrm{MSET}_2(\tilde{\mathcal{G}})}_{2} \,\dot\cup\, \underbrace{\mu \times \tilde{\mathcal{G}} \times \mathrm{MSET}_2\Big(\mathrm{SEQ}^+(\tilde{\mathcal{G}})\Big)}_{3} \,\dot\cup\, \underbrace{\mu \times \tilde{\mathcal{G}} \times \mathrm{SEQ}^+(\tilde{\mathcal{G}})}_{4}\Bigg]. \tag{11}$$

The last component (4) is the one that appears in general galled trees but not in time-consistent galled trees. The generating function satisfies

$$\tilde{\mathcal{G}}(t,u) = t + \frac{1}{2}[\tilde{\mathcal{G}}(t,u)^2 + \tilde{\mathcal{G}}(t^2,u^2)] + \frac{u\tilde{\mathcal{G}}(t,u)}{2}\left[\left(\frac{\tilde{\mathcal{G}}(t,u)}{1-\tilde{\mathcal{G}}(t,u)}\right)^2 + \frac{\tilde{\mathcal{G}}(t^2,u^2)}{1-\tilde{\mathcal{G}}(t^2,u^2)}\right] + \frac{u\tilde{\mathcal{G}}(t,u)^2}{1-\tilde{\mathcal{G}}(t,u)}. \tag{12}$$

3.5 Fixed number of galls

To find the univariate generating function for unlabeled general galled trees with g galls, we must add the gth derivative at $u = 0$ of the last term, $u\tilde{\mathcal{G}}(t,u)^2/[1-\tilde{\mathcal{G}}(t,u)]$, to the univariate generating function for unlabeled time-consistent galled trees [2, eqs. (26)-(29)]. To take the required derivative, writing D_u for $\frac{\partial}{\partial u}$,

we use Leibniz's product rule:

$$\mathrm{D}_u^n(fh) = \sum_{k=0}^{n} \binom{n}{k} (\mathrm{D}_u^k f)(\mathrm{D}_u^{n-k} h). \tag{13}$$

We also use Faà di Bruno's formula for derivatives of a composition:

$$\mathrm{D}_u^m(f\circ h) = \sum_{k_1+2k_2+\ldots+mk_m=m} \frac{m!}{k_1!\,k_2!\,\cdots\,k_m!} \left[\mathrm{D}_x^{k_1+k_2+\ldots+k_m}\left(f(x)\right)\Big|_{x=h(u)}\right]\left[\prod_{\ell=1}^{m}\left(\frac{\mathrm{D}_u^\ell\, h}{\ell!}\right)^{k_\ell}\right]. \tag{14}$$

We use Leibniz's product rule with $f(t,u)=u$, $h(t,u)=\tilde{\mathcal{G}}(t,u)^2/[1-\tilde{\mathcal{G}}(t,u)]$, using g in the role of n (taking the gth derivative); at the value $u=0$, we notice that the only term that is not zero has $k=1$. We use Faà di Bruno's formula with $f(x)=x^2/(1-x)$ and $h(u)=\tilde{\mathcal{G}}(t,u)$, so that when $u=0$, $h(0)=\mathcal{U}(t)$:

$$\begin{aligned}\frac{1}{g!}\,\mathrm{D}_u^g\left(u\frac{\tilde{\mathcal{G}}(t,u)^2}{1-\tilde{\mathcal{G}}(t,u)}\right)\Big|_{u=0} &= \frac{1}{(g-1)!}\,\mathrm{D}_u^{g-1}\left(\frac{\tilde{\mathcal{G}}(t,u)^2}{1-\tilde{\mathcal{G}}(t,u)}\right)\Big|_{u=0}\\ &= \sum_{\substack{k_1+2k_2+\ldots\\ +(g-1)k_{g-1}=g-1}} \frac{1}{k_1!\,k_2!\cdots k_{g-1}!}\,\mathrm{D}_x^{k_1+k_2+\ldots+k_{g-1}}\left(\frac{x^2}{1-x}\right)\Big|_{x=\mathcal{U}(t)} \prod_{\ell=1}^{g-1}\tilde{\mathcal{E}}_\ell(t)^{k_\ell}\end{aligned}$$

We then insert

$$\frac{1}{k!}\,\mathrm{D}_x^k\left(\frac{x^2}{1-x}\right) = \begin{cases} \frac{2x-x^2}{(1-x)^2} = \frac{1}{(1-x)^2}-1 & \text{if } k=1,\\ \frac{1}{(1-x)^{k+1}} & \text{if } k\geq 2.\end{cases}$$

The extra term in $\tilde{\mathcal{E}}_g(t)$ is

$$\sum_{k_1+2k_2+\ldots+(g-1)k_{g-1}=g-1} \binom{k_1+k_2+\ldots+k_{g-1}}{k_1,k_2,\ldots,k_{g-1}} \left(\frac{1}{\left[1-\mathcal{U}(t)\right]^{(\sum_{i=1}^{g-1}k_i)+1}} - \delta_{1,\sum_{i=1}^{g-1}k_i}\right)\prod_{m=1}^{g-1}\tilde{\mathcal{E}}_m(t)^{k_m}.$$

Adding the extra term to the expression in eqs. (26)-(29) of [2], we get

$$\begin{aligned}\tilde{\mathcal{E}}_g(t) = \frac{1}{1-\mathcal{U}(t)}\Bigg[&\Big(\frac{1}{2}\sum_{\ell=1}^{g-1}\tilde{\mathcal{E}}_\ell(t)\,\tilde{\mathcal{E}}_{g-\ell}(t)\Big) + \delta_{0,g \bmod 2}\frac{1}{2}\tilde{\mathcal{E}}_{\frac{g}{2}}(t^2)\\ &+\frac{1}{2}\sum_{k_1+2k_2+\ldots+(g-1)k_{g-1}=g-1}\binom{k_1+k_2+\ldots+k_{g-1}}{k_1,k_2,\ldots,k_{g-1}}\left(\frac{3\mathcal{U}(t)+(\sum_{i=1}^{g-1}k_i)-2}{\left[1-\mathcal{U}(t)\right]^{(\sum_{i=1}^{g-1}k_i)+2}}+\delta_{1,\sum_{i=1}^{g-1}k_i}\right)\prod_{m=1}^{g-1}\tilde{\mathcal{E}}_m(t)^{k_m}\\ &+\frac{1}{2}\sum_{b=1}^{\lfloor\frac{g-1}{2}\rfloor}\tilde{\mathcal{E}}_{(g-2b)-1}(t)\sum_{2r_2+4r_4+\ldots+(2b)r_{2b}=2b}\binom{r_1+r_2+\ldots+r_{2b}}{r_1,r_2,\ldots,r_{2b}}\frac{1}{\left[1-\mathcal{U}(t^2)\right]^{(\sum_{i=1}^{b}r_{2i})+1}}\prod_{m=1}^{b}\tilde{\mathcal{E}}_m(t^2)^{r_{2m}}\\ &+\frac{1}{2}\tilde{\mathcal{E}}_{g-1}(t)\frac{\mathcal{U}(t^2)}{1-\mathcal{U}(t^2)}\\ &+\sum_{k_1+2k_2+\ldots+(g-1)k_{g-1}=g-1}\binom{k_1+k_2+\ldots+k_{g-1}}{k_1,k_2,\ldots,k_{g-1}}\left(\frac{1}{\left[1-\mathcal{U}(t)\right]^{(\sum_{i=1}^{g-1}k_i)+1}}-\delta_{1,\sum_{i=1}^{g-1}k_i}\right)\prod_{m=1}^{g-1}\tilde{\mathcal{E}}_m(t)^{k_m}\Bigg],\end{aligned}$$

or

$$\tilde{\mathcal{E}}_g(t) = \frac{1}{1-\mathcal{U}(t)}\Bigg[\Big(\frac{1}{2}\sum_{\ell=1}^{g-1}\tilde{\mathcal{E}}_\ell(t)\,\tilde{\mathcal{E}}_{g-\ell}(t)\Big) + \delta_{0,g \bmod 2}\frac{1}{2}\tilde{\mathcal{E}}_{\frac{g}{2}}(t^2)$$

$$+ \sum_{k_1+2k_2+\ldots+(g-1)k_{g-1}=g-1} \binom{k_1+k_2+\ldots+k_{g-1}}{k_1,k_2,\ldots,k_{g-1}}$$
$$\times \left(\frac{3\mathcal{U}(t)+(\sum_{i=1}^{g-1} k_i)-2}{2\big[1-\mathcal{U}(t)\big]^{(\sum_{i=1}^{g-1} k_i)+2}} + \frac{1}{\big[1-\mathcal{U}(t)\big]^{(\sum_{i=1}^{g-1} k_i)+1}} - \frac{1}{2}\delta_{1,\sum_{i=1}^{g-1} k_i} \right) \prod_{m=1}^{g-1} \tilde{\mathcal{E}}_m(t)^{k_m}$$
$$+ \frac{1}{2} \sum_{b=1}^{\lfloor \frac{g-1}{2} \rfloor} \tilde{\mathcal{E}}_{(g-2b)-1}(t) \sum_{2r_2+4r_4+\ldots+(2b)r_{2b}=2b} \binom{r_1+r_2+\ldots+r_{2b}}{r_1,r_2,\ldots,r_{2b}} \frac{1}{\big[1-\mathcal{U}(t^2)\big]^{(\sum_{i=1}^{b} r_{2i})+1}} \prod_{m=1}^{b} \tilde{\mathcal{E}}_m(t^2)^{r_{2m}}$$
$$+ \frac{1}{2}\tilde{\mathcal{E}}_{g-1}(t)\frac{\mathcal{U}(t^2)}{1-\mathcal{U}(t^2)}. \tag{15}$$

Writing $\ell = \sum_{i=1}^{g-1} k_i$ and using eq. (3), we notice that $[3\mathcal{U}(t)+\ell-2]/\big[2\big(1-\mathcal{U}(t)\big)^{\ell+2}\big] \sim (\ell+1)/[2\gamma^{\ell+2}(1-t/\rho)^{\ell/2+1}]$ and $1/[1-\mathcal{U}(t)]^{\ell+1} \sim 1/[\gamma^{\ell+1}(1-t/\rho)^{(\ell+1)/2}]$. The term $\delta_{1,\ell}$ is a constant with respect to t and $\ell > 0$. In addition, as shown in Section 3.2, $\tilde{\mathcal{E}}_1(t) \sim \mathcal{E}_1(t)$, and inductively we can conclude that for all m, $0 \le m < g$, $\tilde{\mathcal{E}}_m(t) \sim \mathcal{E}_m(t)$. Therefore, the term unique to the generating function of unlabeled general galled trees does not contribute to the asymptotics of the function. Hence, the asymptotic analysis is identical to that of unlabeled time-consistent galled trees [1, Theorem 10], so that as $n \to \infty$,

$$\tilde{E}_{n,g} = [t^n]\tilde{\mathcal{E}}_g(t) \sim \frac{2^{2g-1}}{(2g)!\,\gamma^{4g-1}\sqrt{\pi}} n^{2g-\frac{3}{2}} \rho^{-n}. \tag{16}$$

Values of $\tilde{E}_{n,g}$ for small n and g are calculated in Section 7.1, and they appear in Table 1.

3.6 Arbitrary number of galls

Fuchs & Gittenberger [12] analyzed unlabeled general galled trees with all possible numbers of galls; we derive the generating function for completion. There are four possible structures (Figure 4): (1) the tree is a single node; (2) it has no root gall and two galled subtrees; (3) it has a root gall in which the sequences leading to the reticulation node are both nonempty, and from all nodes in the root gall, except the root, galled trees descend; (4) it has a root gall in which one of the sequences leading to the reticulation node is empty, and from all nodes in the root gall, except the root, galled trees descend. In symbols,

$$\tilde{\mathcal{A}} = \underbrace{\{\square\}}_{1} \,\dot\cup\, \{\circ\} \times \big[\underbrace{\mathrm{MSET}_2(\tilde{\mathcal{A}})}_{2} \,\dot\cup\, \underbrace{\tilde{\mathcal{A}} \times \mathrm{MSET}_2\big(\mathrm{SEQ}^+(\tilde{\mathcal{A}})\big)}_{3} \,\dot\cup\, \underbrace{\tilde{\mathcal{A}} \times \mathrm{SEQ}^+(\tilde{\mathcal{A}})}_{4} \big]. \tag{17}$$

Converting to a generating function $\tilde{\mathcal{A}} = \sum_{n=0}^{\infty} \tilde{A}_n t^n$,

$$\tilde{\mathcal{A}}(t) = t + \frac{1}{2}[\tilde{\mathcal{A}}(t)^2 + \tilde{\mathcal{A}}(t^2)] + \frac{\tilde{\mathcal{A}}(t)}{2}\left[\left(\frac{\tilde{\mathcal{A}}(t)}{1-\tilde{\mathcal{A}}(t)}\right)^2 + \frac{\tilde{\mathcal{A}}(t^2)}{1-\tilde{\mathcal{A}}(t^2)}\right] + \frac{\tilde{\mathcal{A}}(t)^2}{1-\tilde{\mathcal{A}}(t)}. \tag{18}$$

Eq. (18) matches eq. (28) of [12], which asymptotically showed

$$\tilde{A}_n \sim \frac{0.19659...}{2\sqrt{\pi}} n^{-\frac{3}{2}} (0.11647...)^{-n}. \tag{19}$$

The first values of $\tilde{A}_n$ are calculated in Section 7.1 as the sum $\sum_{g=0}^{n-1} \tilde{E}_{n,g}$. They appear in Table 1.

4 Leaf-labeled general galled trees

The derivations of generating functions for leaf-labeled general galled trees are similar to those of unlabeled general galled trees. The general galled trees have an additional component, compared to time-consistent galled trees, which does not affect the asymptotics.

4.1 No galls

Leaf-labeled general galled trees with no galls are the same as leaf-labeled time-consistent galled trees with no galls. Both are leaf-labeled rooted binary trees. The exponential generating function satisfies [11, p. 129]:

$$\mathfrak{U}(t) = \sum_{n=0}^{\infty} \frac{u_n}{n!} t^n = t + \frac{1}{2}\mathfrak{U}(t)^2, \tag{20}$$

where the u_n count leaf-labeled trees with n leaves. The $\{u_n\}_{n\geq 2}$ satisfy $u_n = (2n-3)!!$ and $u_n = 1$ [9, 10] The factor of $\frac{1}{2}$ in eq. (20) arises because the trees are non-plane. The exponential generating function is

$$\mathfrak{U}(t) = 1 - \sqrt{1-2t}, \tag{21}$$

so that $\frac{1}{2}$ is the radius of convergence. The equation has the same form as the corresponding generating function for unlabeled rooted binary trees as $t \to \rho^-$ (eq. (3)), substituting $\rho = \frac{1}{2}$ and $\gamma = 1$.

To obtain the asymptotic approximation to u_n, as $n \to \infty$, we apply Figure VI.5 of [11], producing

$$u_n \sim \frac{1}{2\sqrt{\pi}} n^{-\frac{3}{2}} \Big(\frac{1}{2}\Big)^{-n} n! \sim \frac{\sqrt{2}}{2} n^{n-1} \Big(\frac{2}{e}\Big)^n, \tag{22}$$

where the last approximation follows by Stirling's approximation to $n!$. The u_n follow OEIS A001147. The first values of u_n appear in Table 2.

4.2 One gall

The structure of leaf-labeled general galled trees follows that of unlabeled general galled trees. From Figure 2, in the notation for the symbolic method for labeled classes [11, p. 148]:

$$\tilde{\mathfrak{E}}_1 = \quad \{\circ\} \times \Bigg[\underbrace{\mathfrak{U} \star \tilde{\mathfrak{E}}_1}_{1} \;\dot\cup\; \underbrace{\tilde{\mathfrak{U}} \star \mathrm{SET}_2\big(\mathrm{SEQ}^+(\tilde{\mathfrak{U}})\big)}_{2} \;\dot\cup\; \underbrace{\tilde{\mathfrak{U}} \star \mathrm{SEQ}^+(\tilde{\mathfrak{U}})}_{3} \Bigg]. \tag{23}$$

The exponential generating function $\tilde{\mathfrak{E}}_1(t) = \sum_{n=0}^{\infty} (\tilde{e}_{n,1}/n!) t^n$ is

$$\begin{aligned} \tilde{\mathfrak{E}}_1(t) &= \tilde{\mathfrak{U}}(t)\, \tilde{\mathfrak{E}}_1(t) + \frac{\tilde{\mathfrak{U}}(t)}{2} \left(\frac{\tilde{\mathfrak{U}}(t)}{1-\tilde{\mathfrak{U}}(t)} \right)^2 + \frac{\tilde{\mathfrak{U}}(t)^2}{1-\tilde{\mathfrak{U}}(t)} \\ &= \frac{\tilde{\mathfrak{U}}(t)^3}{2[1-\tilde{\mathfrak{U}}(t)]^3} + \frac{\tilde{\mathfrak{U}}(t)^2}{[1-\tilde{\mathfrak{U}}(t)]^2}. \end{aligned} \tag{24}$$

Comparing eq. (24) to $\mathfrak{E}_1(t) = \sum_{n=0}^{\infty} (e_{n,1}/n!) t^n$, the exponential generating function of leaf-labeled time-consistent galled trees with one gall (eq. (37) in [2]), eq. (24) has the additional term $\tilde{\mathfrak{U}}(t)^2/[1-\tilde{\mathfrak{U}}(t)]^2$. As in the unlabeled case (Section 3.2), the extra term is not of highest order and hence, as $t \to \frac{1}{2}^-$, $\tilde{\mathfrak{E}}_1(t) \sim \mathfrak{E}_1(t)$.

From eq. (40) in [2], as $n \to \infty$,

$$\tilde{e}_{n,1} = [t^n]\tilde{\mathfrak{E}}_1(t)\, n! \sim \frac{1}{\sqrt{\pi}} n^{\frac{1}{2}} \Big(\frac{1}{2}\Big)^{-n} n! \sim \sqrt{2} n^{n+1} \Big(\frac{2}{e}\Big)^n. \tag{25}$$

The first values of $\tilde{e}_{n,1}$ are calculated in Section 7.2. They appear in Table 2.

4.3 Two galls

As in the case of one gall, the structure of a leaf-labeled general galled tree with two galls follows that of a corresponding unlabeled general galled tree. From Figure 3, we get in the leaf-labeled case:

$$\begin{aligned}\tilde{\mathfrak{E}}_2 = \{\circ\} \times \big[&\underbrace{\mathfrak{U} \star \tilde{\mathfrak{E}}_2}_{1} \;\dot\cup\; \underbrace{\mathrm{SET}_2(\tilde{\mathfrak{E}}_1)}_{2} \;\dot\cup\; \underbrace{\mathfrak{U} \star \big(\mathrm{SEQ}(\mathfrak{U}) \star \tilde{\mathfrak{E}}_1 \star \mathrm{SEQ}(\mathfrak{U})\big) \star \mathrm{SEQ}^+(\mathfrak{U})}_{3} \;\dot\cup\; \underbrace{\tilde{\mathfrak{E}}_1 \star \mathrm{SET}_2\big(\mathrm{SEQ}^+(\mathfrak{U})\big)}_{4} \\ &\dot\cup\; \underbrace{\mathfrak{U} \star \mathrm{SEQ}(\mathfrak{U}) \star \tilde{\mathfrak{E}}_1 \star \mathrm{SEQ}(\mathfrak{U})}_{5} \;\dot\cup\; \underbrace{\tilde{\mathfrak{E}}_1 \star \mathrm{SEQ}^+(\mathfrak{U})}_{6} \big].\end{aligned} \tag{26}$$

Converting to an exponential generating function $\tilde{\mathfrak{E}}_2(t) = \sum_{n=0}^{\infty} (\tilde{e}_{n,2}/n!) t^n$,

$$\tilde{\mathfrak{E}}_2(t) = \mathfrak{U}(t)\, \tilde{\mathfrak{E}}_2(t) + \frac{1}{2}\tilde{\mathfrak{E}}_1(t)^2 + \frac{\mathfrak{U}(t)^2\, \tilde{\mathfrak{E}}_1(t)}{[1-\mathfrak{U}(t)]^3} + \frac{\tilde{\mathfrak{E}}_1(t)}{2}\left(\frac{\mathfrak{U}(t)}{1-\mathfrak{U}(t)}\right)^2 + \frac{\mathfrak{U}(t)\, \tilde{\mathfrak{E}}_1(t)}{[1-\mathfrak{U}(t)]^2} + \frac{\mathfrak{U}(t)\, \tilde{\mathfrak{E}}_1(t)}{1-\mathfrak{U}(t)}.$$

Solving for $\tilde{\mathfrak{E}}_2(t)$, we have

$$\tilde{\mathfrak{E}}_2(t) = \frac{\tilde{\mathfrak{E}}_1(t)^2}{2[1-\mathfrak{U}(t)]} + \frac{\mathfrak{U}(t)^2\, \tilde{\mathfrak{E}}_1(t)}{[1-\mathfrak{U}(t)]^4} + \frac{\mathfrak{U}(t)^2\, \tilde{\mathfrak{E}}_1(t)}{2[1-\mathfrak{U}(t)]^3} + \frac{\mathfrak{U}(t)\, \tilde{\mathfrak{E}}_1(t)}{[1-\mathfrak{U}(t)]^3} + \frac{\mathfrak{U}(t)\, \tilde{\mathfrak{E}}_1(t)}{[1-\mathcal{U}(t)]^2}. \tag{27}$$

In comparison with $\mathfrak{E}_2(t) = \sum_{n=0}^{\infty} (e_{n,2}/n!) t^n$, the exponential generating function for the leaf-labeled time-consistent galled trees with two galls [2, eq. (43)], the generating function for the leaf-labeled general galled trees with two galls contains two terms that arise from cases 5 and 6 in Figure 3. In the same manner as in the unlabeled case (Section 3.3), neither is a term of highest degree. Therefore, asymptotically, as $t \to \frac{1}{2}^-$, $\tilde{\mathfrak{E}}_2(t) \sim \mathfrak{E}_2(t)$.

From eq. (44) of [2], as $n \to \infty$,

$$\tilde{e}_{n,2} = [t^n]\tilde{\mathfrak{E}}_2(\mathfrak{t})\, n! \sim \frac{1}{3\sqrt{\pi}} n^{\frac{5}{2}} \Big(\frac{1}{2}\Big)^{-n} n! \sim \frac{\sqrt{2}}{3} n^{n+3} \Big(\frac{2}{e}\Big)^n. \tag{28}$$

The first values of $\tilde{e}_{n,2}$ are calculated in Section 7.2 and they appear in Table 2.

4.4 Bivariate exponential generating function

We follow the same procedure we used in the unlabeled case and arrive at the exponential generating function that counts leaf-labeled general galled trees with specific numbers of galls. We first find the bivariate exponential generating function $\tilde{\mathfrak{G}}(t,u) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (\tilde{e}_{n,g}/n!) t^n u^m$, counting both leaves and galls. From the structure in Figure 4, the symbolic method yields

$$\tilde{\mathfrak{G}} = \underbrace{\{\square\}}_{1} \;\dot\cup\; \{\circ\} \times \Big[\underbrace{\mathrm{SET}_2(\tilde{\mathfrak{G}})}_{2} \;\dot\cup\; \underbrace{\mu \times \tilde{\mathfrak{G}} \star \mathrm{SET}_2\big(\mathrm{SEQ}^+(\tilde{\mathfrak{G}})\big)}_{3} \;\dot\cup\; \underbrace{\mu \times \tilde{\mathfrak{G}} \star \mathrm{SEQ}^+(\tilde{\mathfrak{G}})}_{4} \Big]. \tag{29}$$

The additional term (component 4) compared to leaf-labeled time-consistent galled trees (eq. (48) of [2]) is similar to that in the unlabeled case (eq. (11)). We therefore have

$$\tilde{\mathfrak{G}}(t,u) = t + \frac{1}{2}\tilde{\mathfrak{G}}(t,u)^2 + \frac{u\tilde{\mathfrak{G}}(t,u)}{2}\left(\frac{\tilde{\mathfrak{G}}(t,u)}{1-\tilde{\mathfrak{G}}(t,u)}\right)^2 + \frac{u\tilde{\mathfrak{G}}(t,u)^2}{1-\tilde{\mathfrak{G}}(t,u)}. \tag{30}$$

4.5 Fixed number of galls

The exponential generating function $\tilde{\mathfrak{E}}_g(t) = \sum_{n=0}^{\infty}(\tilde{e}_{n,g}/n!)t^n$ that counts leaf-labeled general galled trees is the sum of the generating function of leaf-labeled time consistent galled trees $\mathfrak{E}_g(t)$ and the last term in eq. (30), $u\tilde{\mathfrak{G}}(t,u)^2/[1-\tilde{\mathfrak{G}}(t,u)]$. We have already calculated the gth derivative of this extra term at $u=0$ (Section 3.5); hence, combining that calculation (eq. (15)) with eq. (51) of [2], we have

$$\begin{aligned}
\tilde{\mathfrak{E}}_g(t) = \frac{1}{1-\mathfrak{U}(t)}\Bigg[&\frac{1}{2}\sum_{\ell=1}^{g-1}\tilde{\mathfrak{E}}_\ell(t)\,\tilde{\mathfrak{E}}_{g-\ell}(t) \\
&+ \frac{1}{2}\sum_{k_1+2k_2+\ldots+(g-1)k_{g-1}=g-1}\binom{k_1+k_2+\ldots+k_{g-1}}{k_1,k_2,\ldots,k_{g-1}}\left(\frac{3\mathfrak{U}(t)+(\sum_{i=1}^{g-1}k_i)-2}{[1-\mathfrak{U}(t)]^{(\sum_{i=1}^{g-1}k_i)+2}} + \delta_{1,\sum_{i=1}^{g-1}k_i}\right)\prod_{\ell=1}^{g-1}\tilde{\mathfrak{E}}_\ell(t)^{k_\ell} \\
&+ \sum_{k_1+2k_2+\ldots+(g-1)k_{g-1}=g-1}\binom{k_1+k_2+\ldots+k_{g-1}}{k_1,k_2,\ldots,k_{g-1}}\left(\frac{1}{\big[1-\mathfrak{U}(t)\big]^{(\sum_{i=1}^{g-1}k_i)+1}} - \delta_{1,\sum_{i=1}^{g-1}k_i}\right)\prod_{m=1}^{g-1}\tilde{\mathfrak{E}}_m(t)^{k_m}\Bigg],
\end{aligned}$$

or

$$\begin{aligned}
\tilde{\mathfrak{E}}_g(t) = \frac{1}{1-\mathfrak{U}(t)}\Bigg[&\left(\frac{1}{2}\sum_{\ell=1}^{g-1}\tilde{\mathfrak{E}}_\ell(t)\,\tilde{\mathfrak{E}}_{g-\ell}(t)\right) \\
&+ \sum_{k_1+2k_2+\ldots+(g-1)k_{g-1}=g-1}\binom{k_1+k_2+\ldots+k_{g-1}}{k_1,k_2,\ldots,k_{g-1}} \\
&\times\left(\frac{3\mathcal{U}(t)+(\sum_{i=1}^{g-1}k_i)-2}{2\big[1-\mathcal{U}(t)\big]^{(\sum_{i=1}^{g-1}k_i)+2}} + \frac{1}{\big[1-\mathcal{U}(t)\big]^{(\sum_{i=1}^{g-1}k_i)+1}} - \frac{1}{2}\delta_{1,\sum_{i=1}^{g-1}k_i}\right)\prod_{m=1}^{g-1}\tilde{\mathfrak{E}}_m(t)^{k_m}.
\end{aligned} \tag{31}$$

As in the unlabeled case (Section 3.5), we inductively obtain that for all m, $0 \le m < g$, $\tilde{\mathfrak{E}}_m(t) \sim \mathfrak{E}_m(t)$; the extra term in the exponential generating function for leaf-labeled general galled trees compared to leaf-labeled time-consistent galled trees does not contribute to the asymptotics. The asymptotic analysis of $\tilde{\mathfrak{E}}_g(t)$ is identical to that of $\mathfrak{E}_g(t)$ [2, eqs. (52)-(53)]: as $n \to \infty$,

$$\mathfrak{E}_g(t) \sim \frac{(4g-3)!!}{(2g)!\,(1-2t)^{2g-\frac{1}{2}}}, \tag{32}$$

$$\tilde{e}_{n,g} \sim \frac{2^{2g-1}}{(2g)!\sqrt{\pi}}n^{2g-\frac{3}{2}}\left(\frac{1}{2}\right)^{-n}n! \sim \frac{2^{2g-1}\sqrt{2}}{(2g)!}\left(\frac{2}{e}\right)^n n^{n+2g-1}. \tag{33}$$

Values of $\tilde{e}_{n,g}$ for small n and g are calculated in Section 7.2. They appear in Table 2.

4.6 Arbitrary number of galls

As in the case of unlabeled general galled trees (Section 3.6), the analysis of leaf-labeled general galled trees with any possible number of galls has been considered by [12] and earlier by [5]. The tree structure in the

leaf-labeled case follows the unlabeled case (Figure 4), and the symbolic method gives

$$\tilde{\mathfrak{A}} = \underbrace{\{\square\}}_{1} \;\dot\cup\; \{\circ\} \times \big[\underbrace{\mathrm{SET}_2(\tilde{\mathfrak{A}})}_{2} \;\dot\cup\; \underbrace{\tilde{\mathfrak{A}} \star \mathrm{SET}_2\big(\mathrm{SEQ}^+(\tilde{\mathfrak{A}})\big)}_{3} \;\dot\cup\; \underbrace{\tilde{\mathfrak{A}} \star \mathrm{SEQ}^+(\tilde{\mathfrak{A}})}_{4}\big]. \tag{34}$$

Converting to an exponential generating function $\tilde{\mathfrak{A}}(t) = \sum_{n=0}^{\infty}(\tilde{a}_n/n!)t^n$, we have

$$\tilde{\mathfrak{A}}(t) = t + \frac{1}{2}\tilde{\mathfrak{A}}(t)^2 + \frac{\tilde{\mathfrak{A}}(t)}{2}\left(\frac{\tilde{\mathfrak{A}}(t)}{1-\tilde{\mathfrak{A}}(t)}\right)^2 + \frac{\tilde{\mathfrak{A}}(t)^2}{1-\tilde{\mathfrak{A}}(t)}. \tag{35}$$

Eq. (35) is the same as eq. (2) in [12].

From Proposition 6 of [5] and eq. (3) of [12], as $n \to \infty$,

$$\tilde{a}_n \sim \frac{(17-\sqrt{17})\sqrt{2}}{136} n^{n-1}\Big(\frac{8}{e}\Big)^n. \tag{36}$$

The first values of $\tilde{a}_n$ are calculated in Section 7.2 as the sum $\sum_{g=0}^{n-1}\tilde{e}_{n,g}$. They appear in Table 2.

5 Unlabeled simplex time-consistent galled trees

We study the same series of problems for the unlabeled simplex time-consistent galled trees that we examined for unlabeled general galled trees in Section 3. Let $\mathcal{E}_g^1(t) = \sum_{n=0}^{\infty} E_{n,g}^1 t^n$ denote the generating function counting the number of unlabeled simplex galled trees with exactly g galls, with n leaves ($E_{n,g}^1$). As in the case of general galled trees, we examine the special cases of $g = 0$, 1, and 2. We derive a bivariate generating function $\mathcal{G}^1(t,u) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E_{n,m}^1 t^n u^m$, from which we analyze the growth of sequences with any fixed number of galls g. We also derive a generating function $\mathcal{A}^1(t) = \sum_{n=0}^{\infty} A_n^1 t^n$, counting unlabeled simplex time-consistent galled trees with n leaves (A_n^1), summing over all possible numbers of galls.

5.1 No galls

The unlabeled simplex time-consistent galled trees with no galls, are, like the unlabeled time-consistent galled trees with no galls and the unlabeled general galled trees with no galls, simply the unlabeled rooted binary trees studied in Section 3.1.

5.2 One gall

If an unlabeled simplex time-consistent galled tree contains exactly one gall, then as in Section 3.2, we split into cases depending on whether or not the root node is part of the gall (Figure 5). If the root is not part of the gall, then it has two subtrees, a tree with no galls and a simplex time-consistent galled tree with one gall (component 1 in eq. (37) and Figure 5). If the root is part of the gall (component 2), then two non-empty sequences of trees with no galls lead from the root to the reticulation node, from which a single leaf descends, due to the simplex condition. The sequences are non-empty as a result of the time-consistency condition.

In symbolic notation, we get

$$\mathcal{E}_1^1 = \{\circ\} \times \Big[\underbrace{\mathcal{U} \times \mathcal{E}_1^1}_{1} \;\dot\cup\; \underbrace{\{\square\} \times \mathrm{MSET}_2\big(\mathrm{SEQ}^+(\mathcal{U})\big)}_{2}\Big]. \tag{37}$$

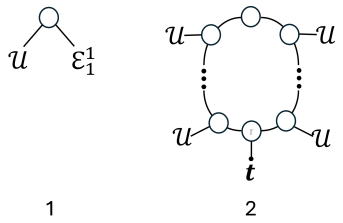

Figure 5: The two structures possible for a (non-plane) unlabeled simplex time-consistent galled tree with exactly one gall.

Converting to a generating function $\mathcal{E}_1^1(t) = \sum_{n=0}^{\infty} E_{n,1}^1 t^n$ yields

$$\mathcal{E}_1^1(t) = \mathcal{U}(t)\,\mathcal{E}_1^1(t) + \frac{t}{2}\left[\left(\frac{\mathcal{U}(t)}{1-\mathcal{U}(t)}\right)^2 + \frac{\mathcal{U}(t^2)}{1-\mathcal{U}(t^2)}\right].$$

Isolating $\mathcal{E}_1^1(t)$ gives

$$\mathcal{E}_1^1(t) = \frac{t\mathcal{U}(t)^2}{2[1-\mathcal{U}(t)]^3} + \frac{t\mathcal{U}(t^2)}{2[1-\mathcal{U}(t)]\,[1-\mathcal{U}(t^2)]}. \tag{38}$$

The $t\mathcal{U}(t)^2/\big(2[1-\mathcal{U}(t)]^3\big)$ term dominates the asymptotic expression for the generating function $\mathcal{E}_1^1(t)$, as $t \to \rho^-$. As $t \to \rho^-$, we have $1-\mathcal{U}(t) \sim \gamma(1-t/\rho)^{\frac{1}{2}}$ (eq. (3)) and $\mathcal{U}(t) \to 1$, so that as $t \to \rho^-$,

$$\mathcal{E}_1^1(t) \sim \frac{\rho}{2\gamma^3(1-\frac{t}{\rho})^{\frac{3}{2}}}. \tag{39}$$

To derive the growth of the coefficients $E_{n,1}^1$, we use the transfer theorems summarized in eq. (1) in Section VII.1 of [11] (p. 441; see also Corollary VI.1, p. 392): if $f(t) = \sum_{n=0}^{\infty} f_n t^n$ for a function f that is Δ-analytic (a technical condition that holds in all our applications), $f(t) \sim (1-t/\nu)^{-\alpha}$ as $t \to \nu$ for $t \in \Delta$, and α is not a non-positive integer, then

$$f_n = [t^n]f(t) \sim \frac{1}{\Gamma(\alpha)} n^{\alpha-1}\nu^{-n}. \tag{40}$$

Using eq. (40) with eq. (39), as $n \to \infty$, we get

$$E_{n,1}^1 = [t^n]\mathcal{E}_1^1(t) \sim \frac{\rho}{2\gamma^3}\frac{n^{\frac{1}{2}}\rho^{-n}}{\Gamma(\frac{3}{2})} = \frac{1}{\gamma^3\sqrt{\pi}} n^{\frac{1}{2}}\rho^{-n+1}, \tag{41}$$

where we note that $\Gamma(\frac{3}{2}) = \sqrt{\pi}/2$. Via eq. (7), the asymptotic expression for $E_{n,1}^1$ satisfies

$$E_{n,1}^1 \sim \rho E_{n,1} \sim \rho\tilde{E}_{n,1}.$$

The first terms of $E_{n,1}^1$ are calculated in Section 7.3. They appear in Table 3.

5.3 Two galls

For two galls, we have three cases (Figure 6). The first two are analogous to the first two cases for general galled trees (Figure 3). If there is no root gall, then as in general galled trees, both galls are contained within the two subtrees of the root node. Either one of the subtrees contains both galls (component 1 in eq. (42)

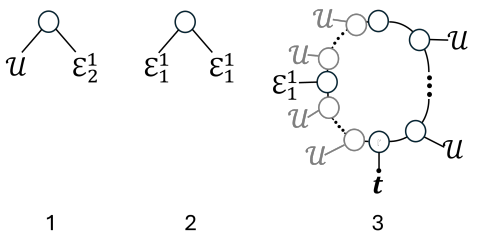

Figure 6: The three structures possible for a (non-plane) unlabeled simplex time-consistent galled tree with exactly two galls.

and Figure 6), or each subtree contains one gall (component 2).

If there is a root gall (component 3 in eq. (42) and Figure 6), then by the time-consistency property, two non-empty sequences of nodes lead from the root node to the reticulation node (from which a leaf descends). A simplex time-consistent galled tree descends from one of the nodes in the sequence, and trees without galls descend from all the remaining nodes. Hence, one of the sequences is a nonempty sequence of trees with no galls and the other is a simplex time-consistent galled tree with one gall, flanked by two (potentially empty) sequences of trees with no galls. In symbolic notation,

$$\mathcal{E}_2^1 = \{\circ\} \times \Big[\underbrace{\mathcal{U} \times \mathcal{E}_2^1}_{1} \;\dot\cup\; \underbrace{\mathrm{MSET}_2(\mathcal{E}_1^1)}_{2} \;\dot\cup\; \underbrace{\{\square\} \times \big(\mathrm{SEQ}(\mathcal{U}) \times \mathcal{E}_1^1 \times \mathrm{SEQ}(\mathcal{U})\big) \times \mathrm{SEQ}^+(\mathcal{U})}_{3}\Big]. \tag{42}$$

Translating to a generating function $\mathcal{E}_2^1(t) = \sum_{n=0}^{\infty} E_{n,2}^1 t^n$, we have

$$\mathcal{E}_2^1(t) = \mathcal{U}(t)\,\mathcal{E}_2^1(t) + \frac{1}{2}[\mathcal{E}_1^1(t)^2 + \mathcal{E}_1^1(t^2)] + t\left(\frac{\mathcal{E}_1^1(t)}{[1-\mathcal{U}(t)]^2}\right)\left(\frac{\mathcal{U}(t)}{1-\mathcal{U}(t)}\right).$$

Isolating $\mathcal{E}_2^1(t)$ gives

$$\mathcal{E}_2^1(t) = \frac{\mathcal{E}_1^1(t)^2}{2[1-\mathcal{U}(t)]} + \frac{\mathcal{E}_1^1(t^2)}{2[1-\mathcal{U}(t)]} + \frac{t\,\mathcal{E}_1^1(t)\,\mathcal{U}(t)}{[1-\mathcal{U}(t)]^4}. \tag{43}$$

To derive asymptotics of $\mathcal{E}_2^1(t)$, note that as $t \to \rho^-$, $\mathcal{E}_1^1(t)$ has order $(1-t/\rho)^{-\frac{3}{2}}$ (eq. (39)) and $1-\mathcal{U}(t)$ has order $(1-t/\rho)^{\frac{1}{2}}$ (eq. (3)). Hence, the first and third terms in eq. (43) have the same order—in particular, $(1-t/\rho)^{-\frac{7}{2}}$—dominating the second term. From the eqs. (3) and (39), as $t \to \rho^-$, we get

$$\mathcal{E}_2^1(t) \sim \frac{\frac{\rho^2}{4\gamma^6(1-\frac{t}{\rho})^3}}{2\gamma(1-\frac{t}{\rho})^{\frac{1}{2}}} + \frac{\rho\frac{\rho}{2\gamma^3(1-\frac{t}{\rho})^{\frac{3}{2}}}}{\gamma^4(1-\frac{t}{\rho})^2} = \frac{5\rho^2}{8\gamma^7(1-\frac{t}{\rho})^{\frac{7}{2}}}. \tag{44}$$

Using eq. (40) with $\alpha = \frac{7}{2}$ this time, as $n \to \infty$,

$$E_{n,2}^1 = [t^n]\mathcal{E}_2^1(t) \sim \frac{5\rho^2}{8\gamma^7}\frac{n^{\frac{5}{2}}\rho^{-n}}{\Gamma(\frac{7}{2})} = \frac{1}{3\gamma^7\sqrt{\pi}} n^{\frac{5}{2}}\rho^{-n+2} \tag{45}$$

where we note that $\Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi}$. Via eq. (10), the asymptotic expression for $E_{n,2}^1$ satisfies

$$E_{n,2}^1 \sim \rho^2 E_{n,2} \sim \rho^2 \tilde{E}_{n,2}.$$

The first terms of $E_{n,2}^1$ are calculated in Section 7.3, and they appear in Table 3.

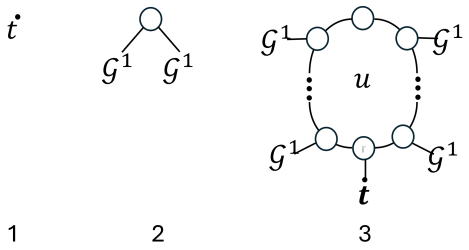

Figure 7: The three structures possible for a (non-plane) unlabeled simplex time-consistent galled tree.

5.4 Bivariate generating function

To derive the generating function $\mathcal{E}_g^1(t) = \sum_{n=0}^{\infty} E_{n,g}^1 t^n$ for any fixed $g \geq 1$, we first turn our attention to the bivariate generating function $\mathcal{G}^1(t,u) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E_{n,m}^1 t^n u^m$, where the coefficient of $t^n u^m$ denotes the number of unlabeled simplex time-consistent galled trees with n leaves and m galls.

A simplex time-consistent galled tree has three possibilities (Figure 7). First, it can be one leaf (component 1 in eq. (46) and Figure 7) or not. Otherwise, if there is no root gall, then the root has two subtrees, each of which is a simplex time-consistent galled tree (component 2). If there is a root gall, then there are two nonempty sequences of nodes leading from the root to the reticulation node, from each of which a simplex time-consistent galled tree can descend (component 3). In the symbolic method construction, we take care to include the leaf descended from the reticulation node and the root gall itself:

$$\mathcal{G}^1 = \underbrace{\{\square\}}_{1} \,\dot\cup\, \{\circ\} \times \Big[\underbrace{\mathrm{MSET}_2(\mathcal{G}^1)}_{2} \,\dot\cup\, \underbrace{\mu \times \{\square\} \times \mathrm{MSET}_2\big(\mathrm{SEQ}^+(\mathcal{G}^1)\big)}_{3}\Big]. \tag{46}$$

Converting to a generating function, we get

$$\mathcal{G}^1(t,u) = t + \frac{1}{2}[\mathcal{G}^1(t,u)^2 + \mathcal{G}^1(t^2,u^2)] + \frac{ut}{2}\left[\left(\frac{\mathcal{G}^1(t,u)}{1-\mathcal{G}^1(t,u)}\right)^2 + \frac{\mathcal{G}^1(t^2,u^2)}{1-\mathcal{G}^1(t^2,u^2)}\right]. \tag{47}$$

5.5 Fixed number of galls

From the bivariate generating function, we derive the generating function $\mathcal{E}_g^1(t)$ counting unlabeled simplex time-consistent galled trees with g galls. As in the case of general galled trees, for each $g \geq 1$,

$$\mathcal{E}_g^1(t) = \frac{1}{g!}\left(\frac{\partial^g}{\partial u^g}\mathcal{G}^1\right)(t,0). \tag{48}$$

To calculate this derivative, we make use of Leibniz's product rule and Faà di Bruno's formula, as in the case of general galled trees (Section 3.5).

We sum the appropriate derivatives of the terms in eq. (47). Derivatives with respect to u of the initial term, t, vanish. For the next two terms, from eqs. (17) and (18) of [2],

$$\frac{1}{2g!}D_u^g[\mathcal{G}(t,u)^2]\big|_{u=0} = \frac{1}{2}\sum_{\ell=0}^{g}\mathcal{E}_\ell^1(t)\,\mathcal{E}_{g-\ell}^1(t) \tag{49}$$

$$\frac{1}{2g!}D_u^g[\mathcal{G}_1(t^2,u^2)]\big|_{u=0} = \frac{1}{2}\mathcal{E}_{\frac{g}{2}}^1(t^2), \tag{50}$$

understanding $\mathcal{E}_{\frac{g}{2}}^1(t^2) = 0$ if g is odd.

The derivative of the next term, $(ut/2)\mathcal{G}^1(t,u)^2/[1-\mathcal{G}^1(t,u)]^2$, is more involved. First, in Leibniz's product rule (eq. (13)), at $u=0$, the only term whose gth derivative does not vanish places one derivative on $\frac{ut}{2}$ and the other $g-1$ derivatives on $\mathcal{G}^1(t,u)^2/[1-\mathcal{G}^1(t,u)]^2$. Hence, we have

$$\frac{1}{g!}D_u^g\left(\frac{ut}{2}\frac{\mathcal{G}^1(t,u)^2}{[1-\mathcal{G}^1(t,u)]^2}\right)\Big|_{u=0}=\frac{t}{2(g-1)!}D_u^{g-1}\left(\frac{\mathcal{G}^1(t,u)^2}{[1-\mathcal{G}^1(t,u)]^2}\right)\Big|_{u=0}.$$

Using Faà di Bruno's formula (eq. (14)) with $f(x)=x^2/(1-x)^2$ and $h(u)=\mathcal{G}^1(t,u)$ (with $u=0$ and $h(0)=\mathcal{U}(t)$) and noting that for any $\ell\geq 1$, $\frac{1}{\ell!}D_u^\ell h(u)|_{u=0}=\mathcal{E}_\ell^1(t)$ (eq. (48)), we can write

$$\frac{1}{g!}D_u^g\left(\frac{ut}{2}\frac{\mathcal{G}^1(t,u)^2}{[1-\mathcal{G}^1(t,u)]^2}\right)\Big|_{u=0}=\frac{t}{2}\sum_{\substack{k_1+2k_2+\dots\\+(g-1)k_{g-1}=g-1}}\frac{1}{k_1!\,k_2!\cdots k_{g-1}!}D_x^{k_1+k_2+\dots+k_{g-1}}\left(\frac{x^2}{(1-x)^2}\right)\Big|_{x=\mathcal{U}(t)}\prod_{\ell=1}^{g-1}\mathcal{E}_\ell^1(t)^{k_\ell}.$$

It is straightforward to show by induction that for any $k\geq 1$,

$$\frac{1}{k!}D_x^k\left[\frac{x^2}{(1-x)^2}\right]=\frac{2x+(k-1)}{(1-x)^{k+2}}.$$

We then have

$$\frac{1}{g!}D_u^g\left(\frac{ut}{2}\frac{\mathcal{G}^1(t,u)^2}{[1-\mathcal{G}^1(t,u)]^2}\right)\Big|_{u=0}=\frac{t}{2}\sum_{\substack{k_1+2k_2+\dots\\+(g-1)k_{g-1}=g-1}}\frac{\left(\sum_{\ell=1}^{g-1}k_\ell\right)!}{k_1!\,k_2!\cdots k_{g-1}!}\frac{2\mathcal{U}(t)+(\sum_{\ell=1}^{g-1}k_\ell)-1}{[1-\mathcal{U}(t)]^{(\sum_{\ell=1}^{g-1}k_\ell)+2}}\prod_{\ell=1}^{g-1}\mathcal{E}_\ell^1(t)^{k_\ell}. \quad (51)$$

For the last term of $G^1(t,u)$, we follow a similar approach. In Leibniz's product rule, at $u=0$, all terms disappear except the term with one derivative of $\frac{ut}{2}$ and the other $g-1$ of $\mathcal{G}^1(t^2,u^2)/[1-\mathcal{G}^1(t^2,u^2)]$. Hence,

$$\frac{1}{g!}D_u^g\left(\frac{ut}{2}\frac{\mathcal{G}^1(t^2,u^2)}{1-\mathcal{G}^1(t^2.u^2)}\right)\Big|_{u=0}=\frac{t}{2(g-1)!}D_u^{g-1}\left(\frac{\mathcal{G}^1(t^2,u^2)}{1-\mathcal{G}^1(t^2.u^2)}\right)\Big|_{u=0}.$$

By eq. (22) of [2], $D_u^{g-1}\big(\mathcal{G}^1(t^2,u^2)/[1-\mathcal{G}^1(t^2,u^2)]\big)\big|_{u=0}\neq 0$ only if $g-1$ is even, and

$$D_u^{g-1}\left(\frac{\mathcal{G}^1(t^2,u^2)}{1-\mathcal{G}^1(t^2,u^2)}\right)\Big|_{u=0}=\sum_{\substack{2r_2+4r_4+\dots\\+(g-1)r_{g-1}=g-1}}\frac{(g-1)!}{r_2!\,r_4!\cdots r_{g-1}!}D_x^{r_1+r_2+\dots+r_{g-1}}\left(\frac{x}{1-x}\right)\Big|_{x=\mathcal{U}(t^2)}\prod_{\ell=1}^{g-1}\frac{1}{\ell!}D_u^\ell\mathcal{G}^1(t^2,u^2)\big|_{u=0}.$$

Now, note that, as we can prove by induction, for each $k\geq 1$,

$$D_x^k\left(\frac{x}{1-x}\right)=\frac{k!}{(1-x)^{k+1}}.$$

In addition, for even m,

$$D_u^m[\mathcal{G}_1(t^2,u^2)]\big|_{u=0}=m!\,\mathcal{E}_{\frac{m}{2}}^1(t^2).$$

Inserting these, we get, for odd g (and even $g-1$),

$$D_u^{g-1}\left(\frac{\mathcal{G}^1(t^2,u^2)}{1-\mathcal{G}^1(t^2,u^2)}\right)\Big|_{u=0}=\sum_{\substack{2r_2+4r_4+\dots\\+(g-1)r_{g-1}=g-1}}\frac{(g-1)!}{r_2!\,r_4!\cdots r_{g-1}!}\left(\frac{\left(\sum_{\ell=1}^{\frac{g-1}{2}}r_{2\ell}\right)!}{[1-\mathcal{U}(t^2)]^{(\sum_{\ell=1}^{(g-1)/2}r_{2\ell})+1}}\right)\prod_{\ell=1}^{\frac{g-1}{2}}\mathcal{E}_1^\ell(t^2)^{k_\ell}. \quad (52)$$

Summing the derivatives of individual terms in eqs. (49), (50), (51), and (52), we get

$$\mathcal{E}_g^1(t) = \frac{1}{2[1-\mathcal{U}(t)]}\Bigg[\Bigg(\sum_{\ell=1}^{g-1}\mathcal{E}_\ell^1(t)\,\mathcal{E}_{g-\ell}^1(t)\Bigg) + \delta_{0,g \bmod 2}\cdot\mathcal{E}_{\frac{g}{2}}^1(t^2)$$
$$+ t\sum_{\substack{k_1+2k_2+\ldots\\+(g-1)k_{g-1}=g-1}}\binom{k_1+k_2+\ldots+k_{g-1}}{k_1,k_2,\ldots,k_{g-1}}\frac{2\mathcal{U}(t)+(\sum_{\ell=1}^{g-1}k_\ell)-1}{[1-\mathcal{U}(t)]^{(\sum_{\ell=1}^{g-1}k_\ell)+2}}\prod_{\ell=1}^{g-1}\mathcal{E}_\ell^1(t)^{k_\ell}$$
$$+ \delta_{1,g \bmod 2}\Bigg(t\sum_{\substack{2r_2+4r_4+\ldots\\+(g-1)r_{g-1}=g-1}}\binom{r_2+r_4+\ldots+r_{g-1}}{r_2,r_4,\ldots,r_{g-1}}\Bigg(\frac{1}{[1-\mathcal{U}(t^2)]^{(\sum_{\ell=1}^{(g-1)/2}r_{2\ell})+1}}\Bigg)\prod_{\ell=1}^{\frac{g-1}{2}}\mathcal{E}_\ell^1(t^2)^{r_\ell}\Bigg)\Bigg]. \tag{53}$$

Using the generating function in eq. (53), we can derive the following result. The asymptotic analysis of $\mathcal{E}_g^1(t)$ largely follows that of $\mathcal{E}_g(t)$ in [1], and by the asymptotic equivalence of $\mathcal{E}_g(t)$ and $\tilde{\mathcal{E}}_g(t)$ in Section 3.5, that of $\tilde{\mathcal{E}}_g(t)$. However, an extra factor of ρ^g appears in the analysis of $\mathcal{E}_g^1(t)$.

Theorem 1 *As $t \to \rho^-$, the generating function $\mathcal{E}_g^1(t)$ for the number of unlabeled simplex time-consistent galled trees with g galls, $g \geq 1$, satisfies*

$$\mathcal{E}_g^1(t) \sim \beta_g \frac{\rho^g}{\gamma^{4g-1}(1-\frac{t}{\rho})^{2g-\frac{1}{2}}} \tag{54}$$

for a constant β_g. Furthermore, for $g \geq 2$, the constants β_g obey the recurrence relation

$$\beta_g = \Bigg(\frac{1}{2}\sum_{\ell=1}^{g-1}\beta_\ell\beta_{g-\ell}\Bigg) + \frac{1}{2}\sum_{\substack{k_1+2k_2+\ldots\\+(g-1)k_{g-1}=g-1}}\binom{k_1+k_2+\ldots+k_{g-1}}{k_1,k_2,\ldots,k_{g-1}}\Bigg[\Bigg(\sum_{\ell=1}^{g-1}k_\ell\Bigg)+1\Bigg]\prod_{\ell=1}^{g-1}\beta_\ell^{k_\ell}, \tag{55}$$

with $\beta_1 = \frac{1}{2}$.

Proof: We perform strong induction on g. The base case of $g = 1$ calculated in eq. (39) follows eq. (54), with $\beta_1 = \frac{1}{2}$. The $g = 2$ case in eq. (44) follows eq. (54), with $\beta_2 = \frac{5}{8} = \frac{1}{2}(\frac{1}{2}\frac{1}{2}) + \frac{1}{2}(1)(2)\frac{1}{2}$ as in eq. (55).

Now, assume that, for some $g \geq 2$, the claim holds for all k, $1 \leq k \leq g-1$. We show that the claim holds for $k = g$ as well. We first determine which terms in eq. (53) contribute to the asymptotic approximation, considering the exponent of $1 - \frac{t}{\rho}$ in the denominator.

Assuming the inductive hypothesis, the second and fourth terms, those involving functions of t^2, do not contribute, as $\rho^2 < \rho$ and ρ^2 is not a singularity of $\mathcal{E}_g^1(t)$. We examine the first and third terms.

For each ℓ, $1 \leq g-1$, by the inductive hypothesis and eq. (3), the exponent of $1 - \frac{t}{\rho}$ in the denominator is $\mathcal{E}_\ell^1(t)\,\mathcal{E}_{g-\ell}^1(t)/[2(1-\mathcal{U}(t))]$ contributes

$$\left(2\ell - \frac{1}{2}\right) + \left[2(g-\ell) - \frac{1}{2}\right] + \frac{1}{2} = 2g - \frac{1}{2}.$$

Similarly, by the inductive hypothesis and eq. (3), each term in the third sum contributes

$$\frac{1}{2}\Bigg[\Bigg(\sum_{\ell=1}^{g-1}k_\ell\Bigg)+3\Bigg] + \sum_{\ell=1}^{g-1}\left(2\ell-\frac{1}{2}\right)k_\ell = 2\Bigg(\sum_{\ell=1}^{g-1}\ell k_\ell\Bigg) + \frac{3}{2} = 2(g-1)+\frac{3}{2} = 2g-\frac{1}{2}.$$

For the exponent of γ in the asymptotic expansion, we note that each factor of $(1-\frac{t}{\rho})^{\frac{1}{2}}$ is accompanied by a factor of γ, so that the exponent of γ in the asymptotic expansion is $2(2g-\frac{1}{2}) = 4g-1$.

For the exponent of ρ, by the inductive hypothesis, for the first term in the asymptotic approximation, $\mathcal{E}^1_\ell(t)\,\mathcal{E}^1_{g-\ell}(t)$ has $\ell + (g-\ell) = g$ as the exponent of ρ. In third term, the exponent of ρ in $\prod_{\ell=1}^{g-1}\mathcal{E}^1_\ell(t)^{k_\ell}$ is $\sum_{\ell=1}^{g-1}\ell k_\ell = g-1$, which, when combined with the t term from outside the sum, as $t\to\rho^-$, leads to an exponent equal to g. We have therefore demonstrated that for $g\geq 1$ galls, each term that contributes to the asymptotic approximation is proportional to $\rho^g/\big[\gamma^{4g-1}(1-\frac{t}{\rho})^{2g-\frac{1}{2}}\big]$.

To compute the proportionality constant, β_g, we note that $\lim_{t\to\rho^-}\mathcal{U}(t) = 1$, and consequently, the first and third terms of eq. (53) produce eq. (55).

□

Proposition 2 *The constants $\{\beta_g\}_{g\geq 1}$ satisfy $2^{2g-1}\beta_g = C_{2g-1}$, where $C_m = \frac{1}{m+1}\binom{2m}{m}$ is a Catalan number.*

Proof: We show an equality between $\mathfrak{D}(t) = \sum_{g=1}^{\infty}2^{2g-1}\beta_g t^{2g-1}$ and the odd part of the generating function of the Catalan numbers $\mathcal{C}_O(t) = \sum_{g=1}^{\infty}C_{2g-1}t^{2g-1}$.

First, following claim 2 in the proof of Proposition 1 of [2], we rewrite $\sum_{\ell=1}^{g-1}k_\ell = \ell$, and we can then replace the sum $\sum_{k_1+2k_2+\ldots+(g-1)k_{g-1}=g-1}\binom{k_1+k_2+\ldots+k_{g-1}}{k_1,k_2,\ldots,k_{g-1}}$ with a sum over compositions $C(g-1,\ell)$ of $g-1$ into ℓ positive integer parts, $\sum_{\ell=1}^{g-1}\sum_{d\in C(g-1,\ell)}$. Eq. (55) becomes

$$\beta_g = \left(\frac{1}{2}\sum_{\ell=1}^{g-1}\beta_\ell\beta_{g-\ell}\right) + \frac{1}{2}\left[(\ell+1)\sum_{\mathbf{d}\in C(g-1,\ell)}\prod_{j=1}^{\ell}\beta_{d_j}\right].$$

Next,

$$\begin{aligned}
\mathfrak{D}(t) &= 2^{2-1}\frac{1}{2}t + \sum_{g=2}^{\infty}\left(\sum_{\ell=1}^{g-1}2^{2g-2}\beta_\ell\beta_{g-\ell}\right)t^{2g-1} + \sum_{g=2}^{\infty}\left[\sum_{\ell=1}^{g-1}(\ell+1)2^{2g-2}\sum_{\mathbf{d}\in C(g-1,\ell)}\prod_{j=1}^{\ell}\beta_{d_j}\right]t^{2g-1}\\
&= t + \left[\sum_{\ell=1}^{\infty}2^{2\ell-1}\beta_\ell t^{2\ell-1}\sum_{g=\ell+1}^{\infty}2^{2(g-\ell)-1}\beta_{g-\ell}t^{2(g-\ell)-1}\right]t\\
&\quad + \left[\sum_{\ell=1}^{\infty}(\ell+1)(2t)^\ell\sum_{g=\ell+1}^{\infty}\sum_{\mathbf{d}\in C(g-1,\ell)}\prod_{j=1}^{\ell}2^{2d_j-1}\beta_{d_j}t^{2d_j-1}\right]t\\
&= t + t\mathfrak{D}(t)^2 + t\sum_{\ell=1}^{\infty}(\ell+1)[2t\mathfrak{D}(t)]^\ell\\
&= t + t\mathfrak{D}(t)^2 + \frac{2t^2\mathfrak{D}(t)}{[1-2t\mathfrak{D}(t)]^2} + \frac{2t^2\mathfrak{D}(t)}{1-2t\mathfrak{D}(t)}.
\end{aligned}$$

We have shown in Proposition 9 of [1] that this implicit expression gives the generating function for the Catalan numbers with odd indices. □

Corollary 3 *The generating function $\mathcal{E}^1_g(t)$ for the number of unlabeled simplex time-consistent galled trees with g galls satisfies as $t\to\rho^-$*

$$\mathcal{E}^1_g(t) \sim \frac{(4g-3)!!\,\rho^g}{(2g)!\,\gamma^{4g-1}(1-\frac{t}{\rho})^{2g-\frac{1}{2}}}. \tag{56}$$

Proof: In Theorem 1, we insert the constant β_g from Proposition 2. The result follows from the identity $C_{2g-1} = [2^{2g-1}(4g-3)!!]/(2g)!$ for the Catalan numbers. □

Using Corollary 3 in conjunction with eq. (40), we obtain the following proposition:

Proposition 4 *The asymptotic growth of the number $E^1_{n,g}$ of unlabeled simplex time-consistent galled trees with n leaves and a fixed number of galls $g \geq 1$ satisfies*

$$E^1_{n,g} \sim \frac{2^{2g-1}}{(2g)!\,\gamma^{4g-1}\sqrt{\pi}} n^{2g-\frac{3}{2}} \rho^{-n+g}. \tag{57}$$

Proof: We apply eq. (40) to the expression in Corollary 3 and simplify using the identity $\Gamma\left(2g-\frac{1}{2}\right) = \sqrt{\pi}[(4g-3)!!]/2^{2g-1}$. □

Via eq. (16), the asymptotic expression for $E^1_{n,g}$ satisfies

$$E^1_{n,g} \sim \rho^g E_{n,g} \sim \rho^g \tilde{E}_{n,g}.$$

Values of $E^1_{n,g}$ with small n and g are calculated in Section 7.3. They appear in Table 3.

5.6 Arbitrary number of galls

Next, we study the generating function of unlabeled simplex time-consistent galled trees with an arbitrary number of galls, $\mathcal{A}^1(t) = \sum_{n=0}^{\infty} A^1_n t^n$. A tree has three possibilities: (1) it is a single node (component 1 in eq. (58) and Figure 7); (2) it has no root gall, and the two subtrees of the root are both simplex time-consistent galled trees (component 2); (3) it has a root gall, from which two nonempty sequences of nodes lead from the root to the reticulation node. From each node in the sequence, a simplex time-consistent galled tree descends (component 3). In symbolic notation,

$$\mathcal{A}^1 = \underbrace{\{\square\}}_{1} \;\dot\cup\; \{\circ\} \times \big[\underbrace{\mathrm{MSET}_2(\mathcal{A}^1)}_{2} \;\dot\cup\; \underbrace{\{\square\} \times \mathrm{MSET}_2\big(\mathrm{SEQ}^+(\mathcal{A}^1)\big)}_{3}\big]. \tag{58}$$

Translating into a generating function, we get

$$\mathcal{A}^1(t) = t + \frac{1}{2}[\mathcal{A}^1(t^2) + \mathcal{A}^1(t)^2] + \frac{t}{2}\left[\left(\frac{\mathcal{A}^1(t)}{1-\mathcal{A}^1(t)}\right)^2 + \frac{\mathcal{A}^1(t^2)}{1-\mathcal{A}^1(t^2)}\right]. \tag{59}$$

To derive asymptotics for $\mathcal{A}^1(t)$ we use the *asymptotics of implicit tree-like classes* theorem [25, 26]. According to the theorem [11, pp. 467-468], if $\mathcal{A}^1(t)$ belongs to the *smooth implicit-function schema*, then as $n \to \infty$, we conclude $A^1_n \sim [\delta/(2\sqrt{\pi})] n^{-\frac{3}{2}} r^{-n}$. In this expression, r is the radius of convergence of $\mathcal{A}^1(t)$ and $\delta = \sqrt{2r\phi_t(r,s)/\phi_{ww}(r,s)}$, where $\phi(t,w) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} S_{n,k} t^n w^k$, $\mathcal{A}^1(t) = \phi\big(t, \mathcal{A}^1(t)\big)$, and (r,s) solves the *characteristic system*

$$s = \phi(r,s) \tag{60}$$

$$1 = \phi_w(r,s). \tag{61}$$

Proposition 5 *$\mathcal{A}^1(t)$ belongs to the smooth implicit-function schema.*

Proof: Showing that $\mathcal{A}^1(t)$ belongs to the smooth implicit-function schema amounts to validating several conditions. (1) $\mathcal{A}^1(t)$ is analytic at 0. (2) $A^1_n \geq 0$ for all $n \geq 0$. (3) $A^1_0 = 0$.

(4) There exists a function $\phi(t,w) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} S_{n,k}t^n w^k$ such that $\mathcal{A}^1(t) = \phi\big(t, \mathcal{A}^1(t)\big)$ and $\phi(t,w)$ satisfies the following conditions:

(a) $\phi(t,w)$ is analytic around $(0,0)$.

(b) The $S_{n,k}$ are nonnegative for all $n,k \geq 0$.

(c) $S_{0,1} \neq 1$.

(d) There exists $S_{n,k} > 0$ for some $k \geq 0$ and $n \geq 2$.

(e) $S_{0,0} = 0$.

(f) There exists a solution (r_0, s_0) to the characteristic system (eqs. (60) and (61)) in the region around $(0,0)$ where ϕ is analytic.

For the first three conditions, (1) $\mathcal{A}^1(t)$ is analytic at 0 from the form of the expression in eq. (59). (2) As A^1_n represents a count of a class of objects, $A^1_n \geq 0$ for all $n \geq 0$. (3) $A^1_0 = 0$ because there are no simplex time-consistent galled trees with no leaves.

(4) We define $\phi(t,w) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} S_{n,k}t^n w^k$ as

$$\phi(t,w) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} S_{n,k}t^n w^k = t + \frac{1}{2}w^2 + \frac{1}{2}\mathcal{A}^1(t^2) + \frac{t}{2}\left[\left(\frac{w}{1-w}\right)^2 + \frac{\mathcal{A}^1(t^2)}{1-\mathcal{A}^1(t^2)}\right],$$

so that $\mathcal{A}^1(t) = \phi\big(t, \mathcal{A}^1(t)\big)$. In addition,

(a) $\phi(t,w)$ is analytic around $(0,0)$ because it is a rational function in w and $\mathcal{A}^1(t)$ has some positive radius of convergence: its coefficients are bounded above by those of $\mathcal{A}(t)$, the generating function for time-consistent galled trees. We have shown that $\mathcal{A}(t)$ has a positive radius of convergence in Section 5.2 of [3], and therefore $\mathcal{A}^1(t)$ does as well.

(b) The $S_{n,k}$ are all nonnegative because $\mathcal{A}^1(t^2) \geq 0$ for all t (and $\mathcal{A}^1(r^2) < 1$, as r^2 is less than the radius of convergence r, $\mathcal{A}^1(t)$ is monotonically increasing for $t < r$, and $\mathcal{A}^1(t) < 1$ for $t < r$).

(c) $S_{0,1} \neq 1$ because $S_{0,1}$ is the coefficient of $t^0 w^1$, and it is equal to 0.

(d) $S_{0,2}$ is the coefficient of $t^0 w^2$ and is equal to $\frac{1}{2}$, so that $S_{0,2} > 0$.

(e) $S_{0,0} = 0$ because $A^1_0 = 0$ so that $\mathcal{A}^1(0) = 0$, leading to $\phi(0,0) = 0$.

(f) There is a solution to the characteristic system as we show below.

We denote $\mathcal{A}^1(r^2) = b$, and the characteristic system is:

$$s = r + \frac{1}{2}s^2 + \frac{1}{2}b + \frac{1}{2}r\left[\left(\frac{s}{1-s}\right)^2 + \frac{b}{1-b}\right] \tag{62}$$

$$1 = s + \frac{rs}{(1-s)^3}. \tag{63}$$

The radius of convergence r is positive; it is less than 1 because the radius of convergence of the generating function of unlabeled simplex time-consistent galled trees is bounded above by the radius of convergence for the generating function of unlabeled binary trees, or $\rho \approx 0.40270$ (Section 3.1). We therefore have

$\mathcal{A}^1(r^2) \le \mathcal{A}^1(r)$. Hence, $\mathcal{A}^1(r^2)$ converges quickly and can be estimated by $\sum_{n=0}^{N} A_n^1 r^{2n}$ for some small N; we use $N = 25$. We calculate $\{A_n^1\}_{n=0}^{N}$ using recursion with $A_0^1 = 0$ and $A_1^1 = 1$ (the trivial tree). For $n \ge 2$,

$$A_n^1 = \frac{1}{2}\left[\left(\sum_{m=1}^{n-1} A_m^1 A_{n-m}^1\right) + \delta_{0,n \bmod 2} A_{\frac{n}{2}}^1 + \left(\sum_{k=3}^{n}(k-2) \sum_{\mathbf{c}\in C(n-1,k-1)} \prod_{i=1}^{k-1} A_{c_i}^1\right) + \delta_{1,n \bmod 2} \sum_{a=1}^{\frac{n-1}{2}} \sum_{\mathbf{c}\in C(\frac{n-1}{2},a)} \prod_{i=1}^{a} A_{c_i}^1\right]. \tag{64}$$

The first two terms of eq. (64), in which the tree does not have a root gall, are equal to those in the recursion for time-consistent galled trees [3, eqs. (15) and (16)]. If a tree has a root gall, then the relevant part in the recursion for time-consistent galled trees is $\frac{1}{2}[\sum_{k=3}^{n}(k-2)\sum_{\mathbf{c}\in C(n,k)}\prod_{i=1}^{k} A_{c_i}] + \frac{1}{2}[\sum_{a=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{\mathbf{c}\in C_p(n,2a+1)}\prod_{i=1}^{a+1} A_{c_i}^1]$. In this expression, k denotes the number of non-root nodes in the root gall, $k-2$ of which can be the reticulation node, keeping in mind the time-consistency condition. The second term is concerned with symmetric trees: a is the number of non-root nodes in the root gall on each side of the reticulation node, and $C_p(a,b)$ is the set of palindromic compositions of a into b positive integers.

To complete the derivation of the terms associated with root galls in eq. (64), we note that in simplex time-consistent galled trees, the child of the reticulation node in the root gall (and any other gall) is a leaf. We must therefore distribute the remaining $n-1$ leaves among the children of the other $k-1$ non-root nodes in the root gall. In addition, because exactly 1 leaf descends from the reticulation node, symmetry between the two sides of the root gall can occur only if n is odd. In symmetric trees, $(n-1)/2$ leaves descend from each side of the reticulation node, and the tree is determined by the children of the nodes in the root gall on one side (say, the left side) of the reticulation node. The values of A_n^1 appear in Table 3.

Returning to the characteristic system, we now have three equations with three unknowns (r, s, b): eq. (62), eq. (63), and $b = \sum_{n=1}^{25} A_n^1 (r^2)^n$. From eq. (63), $r = (1-s)^4/s$. Substituting into eq. (62), we get an equation with only s and b as unknowns. Inserting values $b < 1$ in that equation (because $A_n^1(r^2) < 1$), we numerically solve for s. Plugging these solutions into $r = (1-s)^4/s$, we obtain possible values for r, which we then insert into $\sum_{n=1}^{25} A_n^1 (r^2)^n$. These must coincide with the values of b we used to calculate s. This process gives us $r \approx 0.2344$, $s \approx 0.4349$, and $b \approx 0.0584$. The identification of (r, s) completes the proof that ϕ belongs to the smooth implicit-function schema. □

From the *asymptotics of implicit tree-like classes* theorem, to find the approximation of A_n^1 as $n \to \infty$, we must calculate $\phi_{ww}(r,s)$ and $\phi_t(r,s)$:

$$\phi_{ww}(t,w) = 1 + \frac{t(1+2w)}{(1-w)^4}$$

$$\phi_t(t,w) = 1 + tA^{1'}(t^2) + \frac{w^2}{2(1-w)^2} + \frac{A^1(t^2)}{2[1-A^1(t^2)]} + \frac{t^2 A^{1'}(t^2)}{[1-A^1(t^2)]^2}.$$

We approximate $A^{1'}(r^2)$ by inserting the estimate of r into $\left[\sum_{n=1}^{N} A_n^1(r^2)^n - \sum_{n=1}^{N} A_n^1(r^2-0.001)^n\right]/0.001$. We get $\phi_{ww}(r,s) \approx 5.2993$, $\phi_t(r,s) \approx 1.6716$, and $A^{1'}(r^2) \approx 1.1308$. Finally, as $n \to \infty$, $A_n^1 \sim [\delta/(2\sqrt{\pi})] n^{-\frac{3}{2}} r^{-n}$, with $r \approx 0.2344$ and $\delta \approx 0.3846$.

6 Leaf-labeled simplex time-consistent galled trees

We now move to leaf-labeled simplex time-consistent galled trees. As we observed for general galled trees, the leaf-labeled case is similar to the unlabeled case.

6.1 No galls

Leaf-labeled simplex time-consistent galled trees with no galls are the same as leaf-labeled time-consistent galled trees with no galls and general galled trees with no galls: all are simply the leaf-labeled rooted binary trees covered in Section 4.1.

6.2 One gall

In the case of one gall, using the symbolic method as in Section 5.2 but for labeled classes, we obtain

$$\mathfrak{E}_1^1 = \{\circ\} \times \Big[\underbrace{\mathfrak{U} \star \mathfrak{E}_1^1}_{1} \;\dot\cup\; \underbrace{\{\square\} \star \mathrm{SET}_2\big(\mathrm{SEQ}^+(\mathfrak{U})\big)}_{2} \Big]. \tag{65}$$

The exponential generating function $\mathfrak{E}_1^1(t) = \sum_{n=0}^{\infty} (e_{n,1}^1/n!) t^n$ is

$$\begin{aligned} \mathfrak{E}_1^1(t) &= \mathfrak{U}(t)\, \mathfrak{E}_1^1(t) + \frac{t}{2} \left(\frac{\mathfrak{U}(t)}{1-\mathfrak{U}(t)} \right)^2 \\ &= \frac{t\, \mathfrak{U}(t)^2}{2[1-\mathfrak{U}(t)]^3}. \end{aligned} \tag{66}$$

For the limit, we have, as $t \to \frac{1}{2}^-$,

$$\mathfrak{E}_1^1(t) \sim \frac{1}{4} (1-2t)^{-\frac{3}{2}}. \tag{67}$$

For the asymptotic growth of $e_{n,1}^1$, similarly to the unlabeled case, as $n \to \infty$, by eq. (40), $\gamma(\frac{3}{2}) = \sqrt{\pi}/2$, and the Stirling approximation, we have

$$e_{n,1}^1 = [t^n] \mathfrak{E}_1^1(t)\, n! \sim \frac{1}{4} \frac{n^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} \Big(\frac{1}{2}\Big)^{-n} n! \sim \frac{\sqrt{2}}{2} n^{n+1} \Big(\frac{2}{e}\Big)^n. \tag{68}$$

Comparing to eq. (25), we have $e_{n,1}^1 \sim \frac{1}{2} e_{n,1} \sim \frac{1}{2} \tilde{e}_{n,1}$. The first terms of $e_{n,1}^1$ are calculated in Section 7.4. They appear in Table 4.

6.3 Two galls

For two galls, from the symbolic method, similarly to the unlabeled case (Section 5.3), we get

$$\mathfrak{E}_2^1 = \{\circ\} \times \Big[\underbrace{\mathfrak{U} \star \mathfrak{E}_2^1}_{1} \;\dot\cup\; \underbrace{\mathrm{SET}_2(\mathfrak{E}_1^1)}_{2} \;\dot\cup\; \underbrace{\{\square\} \star \big(\mathrm{SEQ}(\mathfrak{U}) \star \mathfrak{E}_1^1 \star \mathrm{SEQ}(\mathfrak{U})\big) \star \mathrm{SEQ}^+(\mathfrak{U})}_{3} \Big]. \tag{69}$$

The exponential generating function $mathfrakE_2^1(t) = \sum_{n=0}^{\infty} (e_{n,2}^1/n!) t^n$ then satisfies

$$\begin{aligned} \mathfrak{E}_2^1(t) &= \mathfrak{U}(t)\, \mathfrak{E}_2^1(t) + \frac{1}{2} \mathfrak{E}_1^1(t)^2 + t \left(\frac{\mathfrak{E}_1^1(t)}{[1-\mathfrak{U}(t)]^2} \right) \left(\frac{\mathfrak{U}(t)}{1-\mathfrak{U}(t)} \right) \\ &= \frac{\mathfrak{E}_1^1(t)^2}{2[1-\mathfrak{U}(t)]} + \frac{t\, \mathfrak{E}_1^1(t)\, \mathfrak{U}(t)}{[1-\mathfrak{U}(t)]^4}. \end{aligned} \tag{70}$$

As $t \to \frac{1}{2}^-$, following eqs. (21) and (67), we obtain

$$\mathfrak{E}_2^1(t) \sim \frac{5(\frac{1}{2})^2}{8(1-2t)^{\frac{7}{2}}}. \tag{71}$$

Similarly to the unlabeled case, as $n \to \infty$, eq. (40), $\Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi}$, and the Stirling approximation yield

$$e_{n,2}^1 = [t^n]\mathfrak{E}_2^1(t)\, n! \sim \frac{5(\frac{1}{2})^2}{8} \frac{n^{\frac{5}{2}}(\frac{1}{2})^{-n}}{\Gamma(\frac{7}{2})} n! = \frac{1}{3\sqrt{\pi}} n^{\frac{5}{2}} \Big(\frac{1}{2}\Big)^{-n+2} n! \sim \frac{1}{4}\frac{\sqrt{2}}{3} n^{n+3} \Big(\frac{2}{e}\Big)^n. \tag{72}$$

Comparing to eq. (28), we have $e_{n,2}^1 \sim \frac{1}{4} e_{n,2} \sim \frac{1}{4}\tilde{e}_{n,1}$. The first terms of $e_{n,2}^1$ are calculated in Section 7.4. They appear in Table 4.

6.4 Bivariate generating function

Next, to find univariate generating functions $\mathfrak{E}_g^1(t)$ for specific values of g, we examine the bivariate exponential generating function $\mathfrak{G}^1(t,u) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(e_{n,m}^1/n!)t^n u^m$, where $e_{n,m}^1$ denotes the number of leaf-labeled simplex time-consistent galled trees with n leaves and m galls.

For the symbolic method, the tree can just be one node (component 1 in eq. (73) and Figure 7); it can have no root gall and two leaf-labeled simplex time-consistent galled trees descended from the root (component 2), or it can have a root gall and two nonempty sequences of leaf-labeled simplex time-consistent galled trees leading from the root to the reticulation node (component 3). The reticulation node itself has a single leaf as its subtree. In symbolic notation, we have

$$\mathfrak{G}^1 = \underbrace{\{\square\}}_{1} \;\dot\cup\; \{\circ\} \times \big[\underbrace{\mathrm{SET}_2(\mathfrak{G}^1)}_{2} \;\dot\cup\; \underbrace{\mu \times \{\square\} \star \mathrm{SET}_2\big(\mathrm{SEQ}^+(\mathfrak{G}^1)\big)}_{3}\big]. \tag{73}$$

Writing this expression as a generating function, we get

$$\mathfrak{G}^1(t,u) = t + \frac{1}{2}\mathfrak{G}^1(t,u)^2 + \frac{ut}{2}\left(\frac{\mathfrak{G}^1(t,u)}{1-\mathfrak{G}^1(t,u)}\right)^2.$$

6.5 Fixed number of galls

As in the unlabeled case in Section 5.5, we derive the generating function $\mathfrak{E}_g^1(t)$ for leaf-labeled simplex time-consistent galled trees with exactly g galls by noting that, for each $g \geq 1$,

$$\mathfrak{E}_g^1(t) = \frac{1}{g!}\left(\frac{\partial^g}{\partial u^g}\mathfrak{G}^1\right)(t,0).$$

We actually completed most of the steps in the unlabeled case, as the terms in the leaf-labeled case all appear in the bivariate generating function for the unlabeled case; the leaf-labeled case does not have different cases for even and odd g. We copy the appropriate terms from eq. (53):

$$\mathfrak{E}_g^1(t) = \frac{1}{2[1-\mathfrak{U}(t)]}\Bigg[\Bigg(\sum_{\ell=1}^{g-1}\mathfrak{E}_\ell^1(t)\,\mathfrak{E}_{g-\ell}^1(t)\Bigg) + t\sum_{\substack{k_1+2k_2+\ldots\\+(g-1)k_{g-1}=g-1}}\binom{k_1+k_2+\ldots+k_{g-1}}{k_1,k_2,\ldots,k_{g-1}}\frac{2\mathfrak{U}(t)+(\sum_{\ell=1}^{g-1}k_\ell)-1}{[1-\mathfrak{U}(t)]^{(\sum_{\ell=1}^{g-1}k_\ell)+2}}\prod_{\ell=1}^{g-1}\mathfrak{E}_\ell^1(t)^{k_\ell}\Bigg]. \tag{74}$$

As the terms that determine the asymptotic approximation in the unlabeled case are already present in the generating function in the leaf-labeled case, we can immediately deduce the approximation as t approaches the radius of convergence of the generating function, $t \to \frac{1}{2}^-$:

$$\mathfrak{E}_g^1(t) \sim \frac{C_{2g-1}}{2^{2g-1}}\frac{1}{2^g(1-2t)^{2g-\frac{1}{2}}} = \frac{(4g-3)!!}{(2g)!\,2^g(1-2t)^{2g-\frac{1}{2}}}. \tag{75}$$

As $n \to \infty$, by eq. (40) and the Stirling approximation,

$$e_{n,g}^1 \sim \frac{2^{2g-1}}{(2g)!\sqrt{\pi}}n^{2g-\frac{3}{2}}\Big(\frac{1}{2}\Big)^{-n+g}n! \sim \frac{2^{g-1}\sqrt{2}}{(2g)!}n^{n+2g-1}\Big(\frac{2}{e}\Big)^n. \tag{76}$$

Values of $e_{n,g}^1$ for small n and g are calculated in Section 7.4, and they appear in Table 4.

6.6 Arbitrary number of galls

We denote the exponential generating function of leaf-labeled simplex time-consistent galled trees with an arbitrary number of galls by $\mathfrak{A}^1(t) = \sum_{n=0}^\infty (a_n^1/n!)t^n$. The tree has three possibilities: (1) it is trivial with only one node (component 1 in eq. (77) and Figure 7); (2) it has no root gall, and the root node branches into two leaf-labeled simplex time-consistent galled trees (component 2); (3) it has a root gall with two nonempty sequences of nodes from the root to the reticulation node, each with a leaf-labeled simplex time-consistent galled tree as a subtree, and a leaf descends from the reticulation node (component 3). In symbolic notation,

$$\mathfrak{A}^1 = \underbrace{\{\square\}}_{1} \;\dot\cup\; \{\circ\} \times \big[\underbrace{\mathrm{SET}_2(\mathfrak{A}^1)}_{2} \;\dot\cup\; \underbrace{\{\square\}\star\mathrm{SET}_2\big(\mathrm{SEQ}^+(\mathfrak{A}^1)\big)}_{2}\big]. \tag{77}$$

The exponential generating function is

$$\mathfrak{A}^1(t) = t + \frac{1}{2}\mathfrak{A}^1(t)^2 + \frac{t}{2}\Big(\frac{\mathfrak{A}^1(t)}{1-\mathfrak{A}^1(t)}\Big)^2. \tag{78}$$

Following Section 5.6, we find the asymptotic approximation using the asymptotics of implicit tree-like classes theorem.

Proposition 6 *$\mathfrak{A}^1(t)$ belongs to the smooth implicit-function schema.*

Proof: To show that $\mathfrak{A}^1(t)$ belongs to the smooth implicit-function schema we prove the set of conditions set forth in Section 5.6. (1) $\mathfrak{A}^1(t)$ is analytic at 0 from the form of eq. (78). (2) $a_n^1(t) \geq 0$ for all $n \geq 0$, as a count of a class of objects. (3) $a_0^1 = 0$, as no simplex time-consistent galled-trees exist with no leaves.

(4) We define $\psi(t,w) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} s_{n,k}t^n w^k$ as

$$\psi(t,w) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} s_{n,k}t^n w^k = t + \frac{1}{2}w^2 + \frac{t}{2}\left(\frac{w}{1-w}\right)^2,$$

so that $\mathfrak{A}^1(t) = \psi\big(t, \mathfrak{A}^1(t)\big)$. We also have

(a) ψ is analytic around $(0,0)$ because it is a rational function in w and t.

(b) The $s_{n,k}$ are all nonnegative from the definition of ψ.

(c) $s_{0,1} \neq 1$, as it is the coefficient of w, $s_{0,1} = 0$.

(d) $s_{0,2} > 0$, as it is the coefficient of $t^0 w^2$, $s_{0,2} = \frac{1}{2}$.

(e) $s_{0,0} = 0$, because $\psi(0,0) = 0$.

(f) We demonstrate below that the characteristic system has a solution.

The characteristic system is:

$$\omega = \tau + \frac{1}{2}\omega^2 + \frac{\tau}{2}\tau\left(\frac{\omega}{1-\omega}\right)^2, \tag{79}$$

$$1 = \omega + \tau\frac{\omega}{(1-\omega)^3}. \tag{80}$$

From eq. (80), we get $\tau = (1-\omega)^4/\omega$, from which eq. (79) gives us

$$\omega = \frac{3-\sqrt{3}}{3} \approx 0.4226, \tag{81}$$

$$\tau = \frac{3+\sqrt{3}}{18} \approx 0.2629. \tag{82}$$

Therefore, $\mathfrak{A}^1(t)$ belongs to the smooth implicit-function schema. □

According to the *asymptotics of implicit tree-like classes* theorem, as $n \to \infty$, $a_n^1 \sim [\delta/(2\sqrt{\pi})]n^{-\frac{3}{2}}\tau^{-n}n!$, where $\delta = \sqrt{2\tau\psi_t(\tau,\omega)/\psi_{ww}(\tau,\omega)}$. We have $\psi_t(t,w) = 1 + w^2/[2(1-w)^2]$ and $\psi_{ww}(t,w) = 1 + t(1+2w)/(1-w)^4$, so that by Stirling's approximation,

$$a_n^1 \sim \frac{\delta}{2\sqrt{\pi}}n^{-\frac{3}{2}}\tau^{-n}n! \sim \frac{\delta}{\sqrt{2}}n^{n-1}\Big(\frac{\tau^{-1}}{e}\Big)^n, \tag{83}$$

with $\delta = (9-\sqrt{3})\sqrt{3(9+\sqrt{3})}/117 \approx 0.3525$ and $\tau = (3+\sqrt{3})/18 \approx 0.2629$ as in eq. (82).

The first values of a_n^1 are calculated in Section 7.4 as the sum $\sum_{g=0}^{\lfloor\frac{n-1}{2}\rfloor} e_{n,g}^1$. They appear in Table 4.

7 Numerical computations by recursive enumeration

We now compute the exact numbers of phylogenetic networks in the four classes for which we have studied generating functions and asymptotics: unlabeled general galled trees (Section 3), leaf-labeled general galled trees (Section 4), unlabeled simplex time-consistent galled trees (Section 5), and leaf-labeled simplex time-consistent galled trees (Section 6). We perform these computations using recursions.

7.1 Unlabeled general galled trees, $\tilde{E}_{n,g}$

In eqs. (26) and (27) of [3], we calculated a recursion for $E_{n,g}$, the number of unlabeled time-consistent galled trees with n leaves and g galls (eqs. (26) and (27) there). The corresponding number of unlabeled general galled trees $\tilde{E}_{n,g}$ is the sum of $E_{n,g}$ and a term for the additional cases allowed with general galled trees but not time-consistent galled trees.

We then have $\tilde{E}_{1,0} = 1$, and for $n \geq 2$, $0 \leq g \leq n-1$,

$$\begin{aligned}\tilde{E}_{n,g} = \frac{1}{2}\Bigg[&\Bigg(\sum_{\mathbf{c}\in C(n,2)}\sum_{\mathbf{d}\in C(g+2,2)}\prod_{i=1}^{2}\tilde{E}_{c_i,d_i-1}\Bigg) + (\delta_{0,n \bmod 2})(\delta_{0,g \bmod 2})\tilde{E}_{\frac{n}{2},\frac{g}{2}} \\ &+ \Bigg(\sum_{k=3}^{n}(k-2)\sum_{\mathbf{c}\in C(n,k)}\sum_{\mathbf{d}\in C(g-1+k,k)}\prod_{i=1}^{k}\tilde{E}_{c_i,d_i-1}\Bigg) \\ &+ \Bigg(\sum_{a=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{\mathbf{c}\in C_p(n,2a+1)}\sum_{\mathbf{d}\in C_p(g-1+2a+1,2a+1)}\prod_{i=1}^{a+1}\tilde{E}_{c_i,d_i-1}\Bigg)\Bigg] \\ &+ \sum_{k=2}^{n}\sum_{\mathbf{c}\in C(n,k)}\sum_{\mathbf{d}\in C(g-1+k,k)}\prod_{i=1}^{k}\tilde{E}_{c_i,d_i-1}. \end{aligned} \tag{84}$$

In this expression, recall that $C(a,b)$ is the number of compositions of a into b parts, and $C_p(a,b)$ is the number of palindromic compositions. The last term is the term that is additional in relation to the time-consistent case; this term traverses possible numbers k of non-root nodes in the root gall, distributing the n leaves among their children according to a composition $\mathbf{c}$ in the set of compositions of n into k parts, $C(n,k)$. We distribute the remaining $g-1$ galls (one of the g galls is the root gall) across these children according to a composition $\mathbf{d}$ in $C(g-1+k,k)$, where the k added to $g-1$ is due to the technicality that compositions have positive parts, and it is possible for no galls to be descended from a non-root node in the root gall; the added k is deducted in the k appearances of -1 in expressions $\tilde{E}_{c_i,d_i-1}$. In the final term of eq. (84), the position of the reticulation node is specified because the term represents the case in which one of the sequences of nodes from the root of the root gall to the reticulation node is empty.

The number of unlabeled general galled trees with n leaves and any number of galls, $\tilde{A}_n$, is the sum of the numbers of unlabeled general galled trees with n leaves and g galls for all possible numbers of galls, or $\tilde{A}_n = \sum_{g=0}^{n-1}\tilde{E}_{n,g}$. The numerical values in Table 1 suggest a pattern in the numbers of general galled trees with the maximum number of galls.

Proposition 7 *The number of unlabeled general galled trees with n leaves and the maximal number of galls, $n-1$, is a Catalan number, $\tilde{E}_{n,n-1} = C_{n-1}$.*

Proof: We show a bijection between (non-plane) general galled trees with n leaves and $n-1$ galls and plane rooted binary trees with n leaves and $n-1$ internal nodes; the latter is counted by Catalan number C_{n-1} [28, Theorem 1.5.1(ii), p. 8]. First, a general galled tree has $n-1$ galls if and only if all galls have three nodes and all internal nodes are part of galls. The tree contains only 3-node galls and leaves.

The galls are asymmetric: the two descendants of a top node are nodes of two different types, a hybridizing node and a reticulation node. This asymmetry in the gall corresponds to the distinction between left and right descendants of a tree node in an associated plane rooted binary tree. Because the general galled trees we consider are non-plane, we specify, without loss of generality, that the reticulation node is to the left of

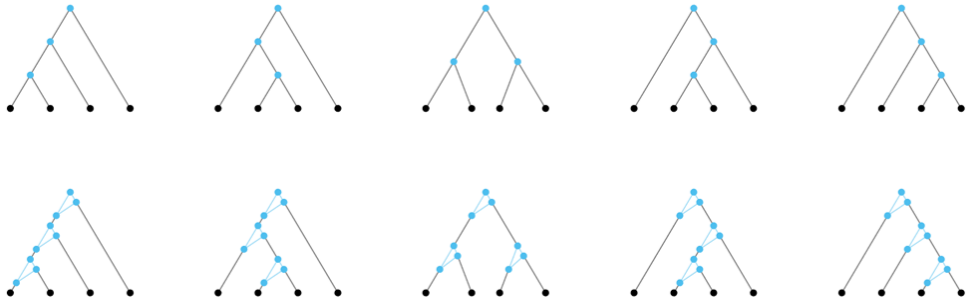

Figure 8: The bijection between plane rooted binary trees with 4 leaves (top) and non-plane general galled trees with 4 leaves and 3 galls (bottom). The bijection matches internal nodes in a plane rooted binary tree with the galls in the corresponding general galled tree (marked in light blue).

the top node in all galls. Therefore, a bijection exists between general galled trees with n leaves and $n-1$ nodes and plane rooted binary trees with $n-1$ internal nodes, in that each gall can be uniquely assigned to an internal node in the binary tree. □

Figure 8 shows an example of the proposition in the case of $n = 4$.

7.2 Leaf-labeled general galled trees, $\tilde{e}_{n,g}$

A recursion for $e_{n,g}$, the number of leaf-labeled time-consistent galled trees with n leaves and g galls appears in eq. (56) of [2]. The number of general galled trees $\tilde{e}_{n,g}$ is obtained by adding a term to that recursion.

We have $\tilde{e}_{1,0} = 1$, and for $n \geq 2$, $0 \leq g \leq n-1$,

$$\begin{aligned}\tilde{e}_{n,g} = \frac{1}{2}\Bigg[&\left(\sum_{m=1}^{n-1}\binom{n}{m}\sum_{\ell=0}^{g}\tilde{e}_{m,\ell}\tilde{e}_{n-m,g-\ell}\right)\\ &+\left(\sum_{k=3}^{n}(k-2)\sum_{\mathbf{c}\in C(n,k)}\sum_{\mathbf{d}\in C(g-1+k,k)}\binom{n}{c_1,c_2,\ldots,c_k}\prod_{i=1}^{k}\tilde{e}_{c_i,d_i-1}\right)\Bigg]\\ &+\left(\sum_{k=2}^{n}\sum_{\mathbf{c}\in C(n,k)}\sum_{\mathbf{d}\in C(g-1+k,k)}\binom{n}{c_1,c_2,\ldots,c_k}\prod_{i=1}^{k}\tilde{e}_{c_i,d_i-1}\right).\end{aligned} \tag{85}$$

The extra term is identical to that in the unlabeled case in eq. (84), with the addition of a multinomial coefficient to account for all possible ways in which the n leaf labels can be distributed across the k subtrees descended from non-root nodes in the root gall.

The number of leaf-labeled general galled trees with n leaves and any number of galls, $\tilde{a}_n$, sums leaf-labeled general galled trees with n leaves and g galls for all possible numbers of galls, or $\tilde{a}_n = \sum_{g=0}^{n-1}\tilde{e}_{n,g}$. We obtain a corollary of Proposition 7 for the leaf-labeled case, verifying a pattern visible in Table 2.

Corollary 8 *The number of leaf-labeled general galled trees with n leaves and the maximal number of galls, $n-1$, satisfies $\tilde{e}_{n,n-1} = C_{n-1}n! = (2n-2)!/(n-1)!$.*

Proof: The result adds leaf labels to the bijectively associated plane rooted binary trees in Proposition 7; the leaf labels are inherited by the associated general galled trees. The number of leaf-labeled plane rooted binary trees with n leaves is $C_{n-1}n!$, so that $\tilde{e}_{n,n-1} = \tilde{E}_{n,n-1}n!$. □

7.3 Unlabeled simplex time-consistent galled trees, $E^1_{n,g}$

For simplex time-consistent galled trees, the recursion is similar to that of time-consistent galled trees (eq. (84). However, because exactly one leaf and zero galls descend from reticulation nodes, $n-1$ leaves are distributed across $k-1$ rather than k subtrees descended from a root gall.

The simplex condition permits symmetry in a root gall only in trees with odd numbers of leaves and odd numbers of galls: one leaf is descended from the reticulation node, and one gall is the root gall. In this case, each side of the reticulation node has $\frac{n-1}{2}$ leaves and $\frac{g-1}{2}$ galls.

The base case of the recursion is $E^1_{1,0}=1$, and for $n\geq 2$, $0\leq g\leq \lfloor\frac{n-1}{2}\rfloor$,

$$\begin{aligned}E^1_{n,g} = \frac{1}{2}\Bigg[&\Bigg(\sum_{\mathbf{c}\in C(n,2)}\sum_{\mathbf{d}\in C(g+2,2)}\prod_{i=1}^{2}E^1_{c_i,d_i-1}\Bigg)+(\delta_{0,n \bmod 2})(\delta_{0,g \bmod 2})E^1_{\frac{n}{2},\frac{g}{2}}\\ &+\Bigg(\sum_{k=3}^{n}(k-2)\sum_{\mathbf{c}\in C(n-1,k-1)}\sum_{\mathbf{d}\in C\left(g-1+(k-1),k-1\right)}\prod_{i=1}^{k-1}E^1_{c_i,d_i-1}\Bigg)\\ &+(\delta_{1,n \bmod 2})(\delta_{1,g \bmod 2})\Bigg(\sum_{a=1}^{\frac{n-1}{2}}\sum_{\mathbf{c}\in C(\frac{n-1}{2},a)}\sum_{\mathbf{d}\in C(\frac{g-1}{2}+a,a)}\prod_{i=1}^{a}E^1_{c_i,d_i-1}\Bigg)\Bigg].\end{aligned} \tag{86}$$

The number of unlabeled simplex time-consistent galled trees with n leaves and any number of galls, A^1_n, is the sum of the numbers of unlabeled simplex time-consistent galled trees with n leaves and g galls for all possible numbers of galls $(0,1,...,\lfloor(n-1)/2\rfloor)$, or $A^1_n=\sum_{g=0}^{\lfloor\frac{n-1}{2}\rfloor}E^1_{n,g}$.

We observe a pattern in the numerical values in Table 3, and verify it in the following proposition.

Proposition 9 *The number of unlabeled simplex time-consistent galled trees with n leaves, n odd, and the maximal number of galls, $\frac{n-1}{2}$, is equal to the number of non-plane unlabeled rooted binary trees with $\frac{n+1}{2}$ leaves, $E^1_{n,(n-1)/2}=U_{(n+1)/2}$.*

Proof: We produce a bijection between unlabeled simplex time-consistent galled trees with the maximal number of galls, $\frac{n-1}{2}$, and unlabeled rooted binary trees. A simplex time-consistent galled tree with the maximal number of galls has $\frac{n-1}{2}$ galls, each with a top node, two hybridizing nodes, and a reticulation node from which a leaf descends. The bijection is between the $\frac{n-1}{2}$ branching events in a non-plane unlabeled rooted binary tree (with $\frac{n+1}{2}$ leaves) and the galls in the unlabeled n-leaf simplex time-consistent galled tree. Each of the $\frac{n-1}{2}$ branching events in the binary tree becomes a 4-node gall by adding two hybridizing nodes on the two edges descending from the tree node and a reticulation node descended from these hybridizing nodes; a leaf descends from the reticulation. The total number of leaves is $\frac{n+1}{2}+\frac{n-1}{2}=n$. □

Figure 9 shows an example with $n=7$, $\frac{n-1}{2}=3$, and $\frac{n+1}{2}=4$.

7.4 Leaf-labeled simplex time-consistent galled trees, $e^1_{n,g}$

In the leaf-labeled case, because each of the n leaf labels is distinct, terms involving symmetric trees do not appear. We incorporate a factor of $n\binom{n-1}{c_1,c_2,\ldots,c_{k-1}}$ to place one label in the leaf descended from the reticulation node of the root gall and to distribute the other $n-1$ leaf labels among the children of the non-root non-reticulation nodes in the root gall.

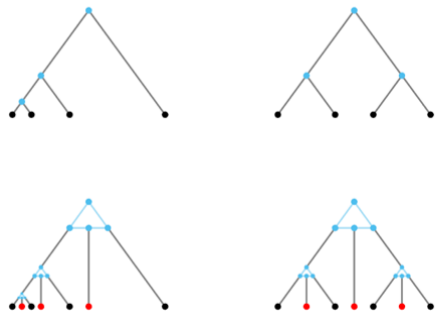

Figure 9: The bijection between non-plane rooted binary trees with 4 leaves (top) and non-plane simplex time-consistent galled trees with 7 leaves and 3 galls (bottom). The bijection matches internal nodes in a rooted binary tree with the galls in the corresponding simplex time-consistent galled tree (marked in light blue). The additional 3 leaves in the galled trees are marked in red.

Therefore, $e^1_{1,0} = 1$, and for $n \geq 2$, $0 \leq g \leq \lfloor \frac{n-1}{2} \rfloor$,

$$e^1_{n,g} = \frac{1}{2}\Bigg[\Bigg(\sum_{m=1}^{n-1}\binom{n}{m}\sum_{\ell=0}^{g} e^1_{m,\ell}e^1_{n-m,g-\ell}\Bigg) + \Bigg(\sum_{k=3}^{n}(k-2)\sum_{\mathbf{c}\in C(n-1,k-1)}\sum_{\mathbf{d}\in C\left(g-1+(k-1),k-1\right)} n\binom{n-1}{c_1,c_2,\ldots,c_{k-1}}\prod_{i=1}^{k-1} e^1_{c_i,d_i-1}\Bigg)\Bigg]. \tag{87}$$

The number of leaf-labeled simplex time-consistent galled trees with n leaves and any number of galls, a^1_n, is the sum of the numbers of leaf-labeled simplex time-consistent galled trees with n leaves and g galls for all possible numbers of galls, or $a^1_n = \sum_{g=0}^{\lfloor \frac{n-1}{2} \rfloor} e^1_{n,g}$.

We write $u_n = (2n-2)!/[2^{n-1}(n-1)!]$ for the number of leaf-labeled trees with n specified labels.

Corollary 10 *The number of leaf-labeled simplex time-consistent galled trees with n leaves, n odd, and the maximal number of galls, $\frac{n-1}{2}$, satisfies $e^1_{n,(n-1)/2} = u_{(n+1)/2}\, n!/(\frac{n+1}{2})!$.*

Proof: By Proposition 9, a bijection exists between leaf-labeled trees with $\frac{n+1}{2}$ leaves, n odd, and leaf-labeled simplex time-consistent galled trees with n leaves and $\frac{n-1}{2}$ galls, with only the $\frac{n+1}{2}$ non-gall descendant leaves labeled. The number of ways of labeling those leaves is $\binom{n}{(n+1)/2}$. The labels of the $\frac{n-1}{2}$ gall descendant leaves can then be assigned in $(\frac{n-1}{2})!$ ways. The number of leaf-labeled simplex time-consistent galled trees with n leaves, n odd, and $\frac{n-1}{2}$ galls is $u_{(n+1)/2}\binom{n}{(n+1)/2}(\frac{n-1}{2})! = u_{(n+1)/2}\, n!/(\frac{n+1}{2})!$. □

8 Discussion

We have studied the combinatorial properties of two types of galled trees, general galled trees and simplex time-consistent galled trees, augmenting our previous studies of a third type of galled trees, the time-consistent galled trees. Our analysis has considered both unlabeled and leaf-labeled cases, and for each class of unlabeled or leaf-labeled galled trees, we have investigated the generating functions and asymptotic approximations for the number of galled trees with n leaves and a fixed number of galls g (Sections 3.1-3.5, 4.1-4.5, 5.1-5.5, and 6.1-6.5), and the generating functions and asymptotics for the number of galled trees

with n leaves, summing across all possible numbers of galls (Sections 3.6, 4.6, 5.6, and 6.6). In Section 7, we examined numerical values for the enumerations for small values of n and g.

For the number of galled trees with n leaves and g galls, as $n \to \infty$, in both the unlabeled and leaf-labeled cases, the class of general galled trees ($\tilde{E}_{n,g}$) has the same asymptotic approximation as the time-consistent galled trees ($E_{n,g}$). The simplex time-consistent galled trees ($E^1_{n,g}$) are asymptotically less numerous in the subexponential term. In the unlabeled case, by eqs. (16) and (57), for $g \geq 1$, we have

$$E^1_{n,g}\rho^{-g} \sim E_{n,g} \sim \tilde{E}_{n,g} \sim \frac{2^{2g-1}}{(2g)!\,\gamma^{4g-1}\sqrt{\pi}} n^{2g-\frac{3}{2}}\rho^{-n}, \tag{88}$$

where $\rho \approx 0.40270$ and $\gamma \approx 1.13003$ are constants that trace originally to the asymptotic analysis of unlabeled trees with no galls, the asymptotic analysis of the Wedderburn–Etherington numbers (eq. (4)).

In the leaf-labeled case, many classes of phylogenetic networks have been studied. Using eq. (1), (2), (33), and (76), and switching from g to k to count reticulations in a more general sense, for $k \geq 1$, we have:

$$\begin{aligned} e^1_{n,k}\frac{(2k)!}{k!} &\sim e_{n,k}\frac{(2k)!}{2^k k!} \sim \tilde{e}_{n,k}\frac{(2k)!}{2^k k!} \\ &\sim PN_{n,k} \sim RV_{n,k} \sim TC_{n,k} \sim N_{n,k} \sim GN_{n,k} \sim GTC_{n,k} \sim \frac{2^{k-1}\sqrt{2}}{k!} n^{n+2k-1}\Big(\frac{2}{e}\Big)^n. \end{aligned} \tag{89}$$

The imposition of a requirement that all reticulations are galls reduces the asymptotic count of the number of general galled trees below that of larger network classes. The simplex condition further reduces this count, as it prevents one subtree from having a further recursively defined structure, fixing that subtree as a leaf.

For the total number of galled trees, not restricting the number of galls, general galled trees possess an extra type of structure not present in time-consistent galled trees: in entry 4 of Figure 4, a top node of a gall can be a parent of the reticulation node. As a result, galls can possess only two nodes other than the root, rather than a minimum of three, and the total number of galls can be as large as $n-1$ rather than $\lfloor \frac{n-1}{2} \rfloor$. Although asymptotic approximations of the numbers of general galled trees, time-consistent galled trees, and simplex time-consistent galled trees with fixed numbers of galls differ only in their subexponential growth in both the unlabeled and leaf-labeled cases, when considering all possible numbers of galls, the exponent also differs. In the unlabeled case, all grow with $[\delta/(2\sqrt{\pi})]n^{-\frac{3}{2}}\alpha^{-n}$ for constants α and δ. The growth is fastest for general galled trees; the approximations for the constants (δ, α) are $(0.1966, 0.1165)$ (eq. (19)); for time-consistent galled trees, $(0.2762, 0.2073)$ (eq. (42) of [3]); for simplex time-consistent galled trees, $(0.3846, 0.2344)$ (Section 5.6). In the leaf-labeled case, the growth follows $(\delta/\sqrt{2})n^{n-1}(\alpha^{-1}/e)^n$, where (δ, α) are approximately $(0.1894, 0.1250)$ for general galled trees (eq. (36)), $(0.2548, 0.2383)$ for time-consistent galled trees [12, eq. (25)], and $(0.3525, 0.2629)$ for simplex time-consistent galled trees (eq. (83)).

We have reported in Tables 1-4 recursive calculations of the exact numbers of unlabeled and leaf-labeled general galled trees and simplex time-consistent galled trees with specific numbers of leaves n and galls g for small numbers of leaves; these tables can be examined together with Tables 4 and 5 of [2], which reports corresponding quantities for time-consistent galled trees. Comparing Tables 2 and 4 to Tables 1 and 3, the tables illustrate the much larger values for leaf-labeled than for unlabeled quantities, comparing Tables 1 and 2 to Tables 3 and 4, they illustrate the faster growth of the general galled trees.

In the case that general galled trees are "saturated" with the maximal number of galls, $n-1$, we found in Proposition 7 that the number of unlabeled general galled trees with n leaves is the Catalan number C_{n-1}. For odd n, we found in Proposition 9 that the number of simplex time-consistent galled trees with the maximal number of galls $\lfloor \frac{n-1}{2} \rfloor$ is simply the number of unlabeled binary rooted trees $U_{(n+1)/2}$. These

results augment corresponding results for other classes of phylogenetic networks with the maximal number of reticulation nodes, including Proposition 2 for tree-child networks in [19], Lemma 2 for galled networks in [18], Theorem 5 for reticulation-visible networks in [7], and Theorem 22 for galled tree-child networks in [8]. Further numerical investigation of the counts of phylogenetic networks may uncover additional bijective constructions involving galled trees or other classes of networks with classic combinatorial structures.

Acknowledgments. Lily Agranat-Tamir, Karthik V. Seetharaman, and Noah A. Rosenberg acknowledge support from National Science Foundation grant DMS-2450005. Michael Fuchs was partially supported by the National Science and Technology Council (NSTC), Taiwan under grant NSTC-113-2115-M-004-004-MY3.

References

[1] L. Agranat-Tamir, M. Fuchs, B. Gittenberger, and N. A. Rosenberg. Asymptotic enumeration of rooted binary unlabeled galled trees with a fixed number of galls. In *35th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms*, volume 302, Article 27 of *Leibniz International Proceedings in Informatics, LIPIcs*. Schloss Dagstuhl—Leibniz-Zentrum für Informatik, Wadern, 2024.

[2] L. Agranat-Tamir, M. Fuchs, B. Gittenberger, and N. A. Rosenberg. Enumerative combinatorics of unlabeled and labeled time-consistent galled trees. *arXiv.2504.16302*, 2025.

[3] L. Agranat-Tamir, S. Mathur, and N. A. Rosenberg. Enumeration of rooted binary unlabeled galled trees. *Bull. Math. Biol.*, 86:45, 2024.

[4] F. Bienvenu, A. Lambert, and M. Steel. Combinatorial and stochastic properties of ranked tree-child networks. *Random Struct. Alg.*, 60:653–689, 2022.

[5] M. Bouvel, P. Gambette, and M. Mansouri. Counting phylogenetic networks of level 1 and 2. *J. Math. Biol.*, 81:1357–1395, 2020.

[6] G. Cardona and L. Zhang. Counting and enumerating tree-child networks and their subclasses. *J. Comp. System Sci.*, 114:84–104, 2020.

[7] Y.-S. Chang and M. Fuchs. Counting phylogenetic networks with few reticulation vertices: galled and reticulation-visible networks. *Bull. Math. Biol.*, 86:76, 2024.

[8] Y.-S. Chang, M. Fuchs, and G.-R. Yu. Galled tree-child networks. In *35th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms*, volume 302, Article 2 of *Leibniz International Proceedings in Informatics, LIPIcs*. Schloss Dagstuhl—Leibniz-Zentrum für Informatik, Wadern, 2024.

[9] A. W. F. Edwards and L. L. Cavalli-Sforza. Reconstruction of evolutionary trees. In V. H. Heywood and J. McNeil, editors, *Phenetic and Phylogenetic Classification*, pages 67–76, London, 1964. Systematics Association.

[10] J. Felsenstein. The number of evolutionary trees. *Syst. Zool.*, 27:27–33, 1978.

[11] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, Cambridge, UK, 2009.

[12] M. Fuchs and B. Gittenberger. Sackin indices for labeled and unlabeled classes of galled trees. *J. Math. Biol.*, 90:42, 2025.

[13] M. Fuchs, B. Gittenberger, and M. Mansouri. Counting phylogenetic networks with few reticulation vertices: tree-child and normal networks. *Australas J. Combin.*, 73:385–423, 2019.

[14] M. Fuchs, B. Gittenberger, and M. Mansouri. Counting phylogenetic networks with few reticulation vertices: exact enumeration and corrections. *Australas J. Combin.*, 89:257–282, 2021.

[15] M. Fuchs, E.-Y Huang, and G.-R. Yu. Counting phylogenetic networks with few reticulation vertices: a second approach. *Discr. Appl. Math.*, 320:140–149, 2022.

[16] M. Fuchs and M. Steel. A dichotomy law for certain classes of phylogenetic networks. *Bull. Math. Biol.*, 87:153, 2025.

[17] M. Fuchs, M. Steel, and Q. Zhang. Asymptotic enumeration of normal and hybridization networks via tree decoration. *Bull. Math. Biol.*, 87:69, 2025.

[18] M. Fuchs, G.-R. Yu, and L. Zhang. Asymptotic enumeration and distributional properties of galled networks. *J. Comb. Theory Ser. A*, 189:105599, 2022.

[19] M. Fuchs, G.-R. Yu, and Louxin Zhang. On the asymptotic growth of the number of tree-child networks. *Eur. J. Combin.*, 93:103278, 2021.

[20] A. D. M. Gunawan, J. Rathin, and L. Zhang. Counting and enumerating galled networks. *Discr. Appl. Math.*, 283:644–654, 2020.

[21] E. F. Harding. The probabilities of rooted tree-shapes generated by random bifurcation. *Adv. Appl. Prob.*, 3:44–77, 1971.

[22] S. Kong, J. C. Pons, L. Kubatko, and K. Wicke. Classes of explicit phylogenetic networks and their biological and mathematical significance. *J. Math. Biol.*, 84:47, 2022.

[23] M. Mansouri. Counting general phylogenetic networks. *Australas. J. Combin.*, 83:40–86, 2022.

[24] S. Mathur and N. A. Rosenberg. All galls are divided into three or more parts: recursive enumeration of labeled histories for galled trees. *Alg. Mol. Biol.*, 18:1, 2023.

[25] A. Meir and J. W. Moon. Erratum: "On an asymptotic method in enumeration". *J. Comb. Theory, Ser. A*, 52:163, 1989.

[26] A. Meir and J. W. Moon. On an asymptotic method in enumeration. *J. Comb. Theory, Ser. A*, 51:77–89, 1989.

[27] R. Otter. The number of trees. *Ann. Math.*, 49:583–599, 1948.

[28] R. P. Stanley. *Catalan Numbers*. Cambridge University Press, Cambridge, 2015.

[29] L. Zhang. Generating normal networks via leaf insertion and nearest neighbor interchange. *BMC Bioinformatics*, 20:642, 2019.

[30] L. Zhang. The Sackin index of simplex networks. In L. Jin and D. Durand, editors, *Comparative Genomics: 19th International Conference, RECOMB-CG 2022, La Jolla, CA, USA, May 20–21, 2022, Proceedings*, pages 52–67, Berlin, 2022. Springer.

Number of leaves (n)	Total number of trees ($\tilde{A}_n$)	Number of trees with a fixed number of galls ($\tilde{E}_{n,g}$)											
		$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$	$g=6$	$g=7$	$g=8$	$g=9$	$g=10$	$g=11$
1	1	1	-	-	-	-	-	-	-	-	-	-	-
2	2	1	1	-	-	-	-	-	-	-	-	-	-
3	8	1	5	2	-	-	-	-	-	-	-	-	-
4	43	2	16	20	5	-	-	-	-	-	-	-	-
5	255	3	49	113	76	14	-	-	-	-	-	-	-
6	1,637	6	140	526	634	289	42	-	-	-	-	-	-
7	11,004	11	392	2,143	4,030	3,198	1,098	132	-	-	-	-	-
8	76,634	23	1,072	8,076	21,604	26,024	15,217	4,189	429	-	-	-	-
9	547,539	46	2,898	28,667	103,267	173,886	151,509	69,808	16,028	1,430	-	-	-
10	3,992,150	98	7,744	97,498	453,712	1,013,118	1,217,115	824,004	312,468	61,531	4,862	-	-
11	29,579,670	207	20,538	320,388	1,869,315	5,328,525	8,392,643	7,749,108	4,271,018	1,374,266	236,866	16,796	-
12	222,089,534	451	54,095	1,024,750	7,318,842	25,886,758	51,562,707	61,852,619	46,085,573	21,365,251	5,965,532	914,170	58,786

Table 1: Numbers $\tilde{E}_{n,g}$ of small unlabeled general galled trees with specified numbers of leaves (n) and galls (g). Entries $\tilde{E}_{n,g}$ are computed recursively from eq. (84) in Section 7.1. For $g=0$, $E_{n,0}=U_n$.

Number of leaves (n)	Total number of trees ($\tilde{a}_n$)	Number of trees with a fixed number of galls ($\tilde{e}_{n,g}$)											
		$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$	$g=6$	$g=7$	$g=8$	$g=9$	$g=10$	$g=11$
1	1	1	-	-	-	-	-	-	-	-	-	-	-
2	3	1	2	-	-	-	-	-	-	-	-	-	-
3	36	3	21	12	-	-	-	-	-	-	-	-	-
4	723	15	228	360	120	-	-	-	-	-	-	-	-
5	20,280	105	2,805	8550	7,140	1,680	-	-	-	-	-	-	-
6	730,755	945	39,330	196,560	297,360	166,320	30,240	-	-	-	-	-	-
7	32,171,580	10,395	623,385	4,639,320	11,007,360	10,735,200	4,490,640	665,280	-	-	-	-	-
8	1,673,573,895	135,135	11,055,240	114,896,880	392,893,200	583,783,200	415,134,720	138,378,240	17,297,280	-	-	-	-
9	100,442,870,640	2,027,025	217,237,545	3,010,855,050	14,023,894,500	29,357,559,000	31,167,536,400	17,344,847,520	4,799,995,200	518,918,400	-	-	-
10	6,831,585,584,775	34,459,425	4,689,345,150	83,706,777,000	509,858,533,800	1,427,560,734,600	2,100,224,926,800	1,719,046,929,600	783,566,784,000	185,253,868,800	17,643,225,600	-	-
11	519,288,366,989,700	654,729,075	110,367,613,125	2,469,131,626,500	19,067,559,310,800	68,780,493,381,600	133,494,142,181,400	149,527,115,337,600	99,101,998,195,200	38,188,761,811,200	7,877,700,230,400	670,442,572,800	-
12	43,626,178,967,384,480	13,749,310,575	2,813,814,441,900	77,183,193,087,000	737,459,213,891,400	3,330,980,344,692,000	8,233,128,718,214,400	12,036,684,871,411,200	10,810,119,006,086,400	6,002,304,743,635,200	2,001,271,079,808,000	366,061,644,748,800	28,158,588,057,600

Table 2: Numbers $\tilde{e}_{n,g}$ of small leaf-labeled general galled trees with specified numbers of leaves (n) and galls (g). Entries $\tilde{e}_{n,g}$ are computed recursively from eq. (85) in Section 7.2. For $g=0$, $\tilde{e}_{n,0}$ is the number of leaf-labeled rooted binary trees $(2n-3)!! = (2n-2)!/[2^{n-1}(n-1)!]$.

Number of leaves (n)	Total number of trees (A_n^1)	Number of trees with a fixed number of galls ($E_{n,g}^1$)							
		$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$	$g=6$	$g=7$
1	1	1	-	-	-	-	-	-	-
2	1	1	-	-	-	-	-	-	-
3	2	1	1	-	-	-	-	-	-
4	5	2	3	-	-	-	-	-	-
5	15	3	11	1	-	-	-	-	-
6	47	6	32	9	-	-	-	-	-
7	158	11	96	49	2	-	-	-	-
8	545	23	270	230	22	-	-	-	-
9	1,940	46	758	951	182	3	-	-	-
10	7,030	98	2,074	3,657	1,146	55	-	-	-
11	25,917	207	5,638	13,242	6,235	589	6	-	-
12	96,755	451	15,132	45,955	30,295	4,789	133	-	-
13	365,274	983	40,364	153,943	135,839	32,331	1,803	11	-
14	1,391,629	2,179	106,858	501,536	571,309	191,543	17,881	323	-
15	5,344,385	4,850	281,494	1,595,983	2,284,292	1,027,307	145,178	5,258	23

Table 3: Numbers $E_{n,g}^1$ of small unlabeled simplex time-consistent galled trees with specified numbers of leaves (n) and galls (g). Entries $E_{n,g}^1$ are computed recursively from eq. (86) in Section 7.3. The values of $\tilde{A}_n$ for $n \leq 25$ missing from the table, as used in Section 5.6, are calculated recursively using eq. (64): $A_{16}^1 = 20,665,633$, $A_{17}^1 = 80,394,281$, $A_{18}^1 = 314,422,685$, $A_{19}^1 = 1,235,563,322$, $A_{20}^1 = 4,875,984,643$, $A_{21}^1 = 19,316,320,501$, $A_{22}^1 = 76,788,144,951$, $A_{23}^1 = 306,222,020,733$, $A_{24}^1 = 1,224,709,419,623$, $A_{25}^1 = 4,911,122,651,176$.

Number of leaves (n)	Total number of trees (a_n^1)	Number of trees with a fixed number of galls ($e_{n,g}^1$)							
		$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$	$g=6$	$g=7$
1	1	1	-	-	-	-	-	-	-
2	1	1	-	-	-	-	-	-	-
3	6	3	3	-	-	-	-	-	-
4	63	15	48	-	-	-	-	-	-
5	870	105	705	60	-	-	-	-	-
6	15,075	945	10,980	3,150	-	-	-	-	-
7	317,520	10,395	186,795	117,180	3,150	-	-	-	-
8	7,891,695	135,135	3,487,680	3,916,080	352,800	-	-	-	-
9	226,068,570	2,027,025	71,291,745	127,348,200	25,084,080	317,520	-	-	-
10	7,332,968,475	34,459,425	1,587,996,900	4,173,488,550	1,475,107,200	61,916,400	-	-	-
11	265,668,133,500	654,729,075	38,347,414,875	140,238,945,000	79,152,207,750	7,222,446,000	52,390,800	-	-
12	10,633,098,774,375	13,749,310,575	998,905,446,000	4,875,076,206,000	4,065,788,034,000	663,772,725,000	15,807,052,800	-	-
13	465,947,254,322,550	316,234,143,225	27,937,693,008,225	176,135,053,594,500	205,330,075,750,800	53,465,297,886,000	2,750,056,709,400	12,843,230,400	-
14	22,187,548,315,807,876	7,905,853,580,625	835,382,740,292,100	6,628,818,891,444,750	10,359,374,548,723,200	3,985,285,430,926,800	365,229,364,500,000	5,551,486,340,400	-
15	1,140,812,140,619,151,104	213,458,046,676,875	26,603,123,656,984,876	260,094,334,918,918,496	527,417,382,642,900,800	283,763,693,046,833,984	41,365,877,823,270,000	1,349,887,731,192,000	4,382,752,374,000

Table 4: Numbers $e_{n,g}^1$ of small leaf-labeled simplex time-consistent galled trees with specified numbers of leaves (n) and galls (g). Entries $e_{n,g}^1$ are computed recursively from eq. (87) in Section 7.4.