ON MAXIMA IN GEOMETRIC WORDS THAT SATISFY A GENERALIZED RESTRICTED GROWTH PROPERTY

(joint work with Mehri Javanian)

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Geometric Words

Words: $\omega_1 \cdots \omega_n$ with $\omega_i \in \mathbb{N}$.
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Random model:

$\omega_i$ are independent and geometrically distributed with success probability $p$, i.e.,

$$\mathbb{P}(\omega_i = \ell) = pq^{\ell-1}, \quad \ell \in \mathbb{N},$$

where $q := 1 - p$. 
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Studied because related to:

- Approximate counting;
- Digital trees
(Some) Previous Work

Left-to-right maxima

Maximum value
Bruss & O'Cinneide (1990); Eisenberg (2008); Prodinger (2012)

# of times the maximum occurs
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Other parameters:
# of different letters, missing letters, gaps, inversion, ascends and descends, runs, etc.
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\[ L_n = \# \text{ of left-to-right maxima of a geometric word.} \]
\[ Q := 1/q. \]
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$Q := 1/q.$

Theorem (Prodinger; 1996)

We have,

$$
\mathbb{E}(L_n) \sim p \log_Q n + \Phi_1(\log_Q n) \quad \text{and} \quad \text{Var}(L_n) \sim pq \log_Q n + \Phi_2(\log_Q n),
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where $\Phi_1, \Phi_2$ are 1-periodic functions.
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**Theorem (Bai & Hwang & Liang; 1998)**

We have,

\[
\frac{L_n - p \log_q n}{\sqrt{pq \log_q n}} \xrightarrow{d} N(0, 1).
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Geometric Words satisfying GRGP

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Geometric Words satisfying GRGP

\( \omega = \omega_1 \cdots \omega_n. \)

\( \omega \) satisfies **generalized restricted growth property** (GRGP):

\[ \omega_i \leq d + \max\{\omega_0, \ldots, \omega_{i-1}\} \quad \text{with} \quad \omega_0 := 0 \]

and \( d \geq 1. \)
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\( d = 1 \):

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- \( L^{(1)}_n = \text{maximum value} = \# \text{ of blocks} \)
Distribution of $L^{(d)}_n$

We have,

$$p_{n+1,k} = \sum_{\ell=1}^d pq_{\ell-1} \sum_{j=0}^n \binom{n}{j} (1-q_{\ell})^{n-j} q_{\ell j}^{p_{j,k}-1}.$$ 

Then,

$$P(L^{(d)}_n = k) = \frac{p_{n,k}}{\sum_{k} p_{n,k}}.$$ 

Goal: find asymptotics of moment generating function

$$E(e^{t L^{(d)}_n}) = \sum_{k} P(L^{(d)}_n = k) e^{kt}$$

in a complex neighbourhood of 0.
Distribution of $L_n^{(d)}$

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$$\mathbb{P} \left( L_n^{(d)} = k \right) = \frac{p_{n,k}}{\sum_k p_{n,k}}.$$
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$p_{n,k} =$ probability that a word satisfying GRGP has $k$ left-to-right maxima

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**Goal**: find asymptotics of moment generating function

$$\mathbb{E} \left( e^{L_n^{(d)} t} \right) = \sum_k \mathbb{P} \left( L_n^{(d)} = k \right) e^{kt}$$

in a complex neighbourhood of 0.
Asymptotics of Moment Generating Function (i)

Set

\[ \tilde{L}(z, t) := e^{-z} \sum_{n} \sum_{k} p_{n,k} e^{kt} \frac{z^n}{n!}. \]
Asymptotics of Moment Generating Function (i)

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\[ \tilde{L}(z, t) := e^{-z} \sum_n \sum_k p_{n,k} e^{kt} \frac{z^n}{n!}. \]

Then,

\[ \tilde{L}(z, t) + \frac{\partial}{\partial z} \tilde{L}(z, t) = pe^t \sum_{\ell=1}^d q^{\ell-1} \tilde{L}(q^\ell z, t). \]
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Can be solved with the **Mellin transform**:

\[ \mathcal{M}[\tilde{f}(z); \omega] := \int_0^\infty \tilde{f}(z) z^{\omega-1} dz \]

because of
\[ \mathcal{M}[\tilde{f}(az); \omega] = a^{-\omega} \mathcal{M}[\tilde{f}(z); \omega]. \]
Converse Mapping Theorem

Theorem (Flajolet, Gourdon, Dumas; 1995)

Let the Mellin transform of $\tilde{f}(z)$ exist in the strip $\langle \alpha, \beta \rangle$.

Assume that $\mathcal{M}[\tilde{f}(z); s]$ can be continued to a meromorphic function on $\langle \alpha, \gamma \rangle$ with $\beta < \gamma$ with simple poles at $s_1, \cdots, s_k$.

Then, under some technical conditions,

$$\tilde{f}(z) = - \sum_{j=1}^{k} \text{Res}(\mathcal{M}[\tilde{f}(z); s], s = s_j) z^{-s_j} + O(z^{-\gamma})$$

as $z \to \infty$. 
Applying the converse mapping theorem gives:

\[ \tilde{L}(z, t) \sim -\frac{P_t(1)\Omega_t(1)}{\log(Q)\rho_t P'_t(\rho_t)\Omega_t(\rho_t)} z^{-\log Q \rho_t} \sum_k \Gamma(\log Q \rho_t + \chi_k) z^{-\chi_k}, \]

where \( \chi_k = \frac{2k\pi i}{\log(Q)} \) and \( P_t(z) = 1 - pe^{td} \sum_{\ell=1} q^\ell - 1 z^\ell, \)

\( \Omega_t(s) = \prod_{\ell \geq 1} P_t(q_s), \)

and \( \rho_t \) is the unique positive root of \( P_t(z) \).

Finally, note that \( \tilde{L}(n, t) = e^{-n} \sum_{m} \sum_{k} p_{m,k} e^{kt} n^m \approx \sum_{k} p_{n,k} e^{kt} \) ... Poisson heuristic
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Asymptotics of Moment Generating Function (ii)

Applying the converse mapping theorem gives:

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Finally, note that

$$\tilde{L}(n, t) = e^{-n} \sum_m \sum_k p_{m,k} e^{kt} \frac{n^m}{m!} \sim \sum_k p_{n,k} e^{kt} \ldots \text{Poisson heuristic!}$$
Proposition

*Uniformly in a neighbourhood of 0,*

\[
\mathbb{E} \left( e^{L_n^{(d)} t} \right) \sim \frac{P_t(1) \Omega_t(1) \rho_0 P'_0(\rho_0) \Omega_0(\rho_0)}{q^d \Omega_0(1) \rho_t P'_t(\rho_t) \Omega_t(\rho_t)} n^{-\log Q(\rho_t/\rho_0)} \\
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**Corollary**

*For* \( m \geq 1, *

\[
\mathbb{E}(L_n^{(1)} - \log_Q n)^m \sim \Phi_m^{(1)}(\log_Q n),
\]

*where* \( \Phi_m^{(1)} *are 1-periodic functions.*
Theorem

$L_n^{(1)} - \log Q_n$ does not converge to a fixed limit law.
Limit Law of $L_n^{(d)}$

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We have,

$$
L_n^{(d)} + \frac{\log Q_n / (\rho_0 P'_0(\rho_0))}{\sqrt{\log Q_n}} \xrightarrow{d} N(0, \sigma^2_d),
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for a constant $\sigma^2_d$ which is $> 0$ iff $d \geq 2$. 
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We have,

$$L_n^{(d)} + \frac{\log Q n / \left( \rho_0 P'_0(\rho_0) \right)}{\sqrt{\log Q n}} \xrightarrow{d} N(0, \sigma_d^2),$$

for a constant $\sigma_d^2$ which is $> 0$ iff $d \geq 2$.

Thus, the limit law of $L_n^{(d)}$ undergoes a phase change from non-existence for $d = 1$ to normal for $d \geq 2$!
Maximum Value (i)

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**Question**: does the limit law of \( M_n^{(d)} \) also undergo a phase change?
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**Theorem**

*For* \( m \geq 1 \),

\[ \mathbb{E}(M_n^{(d)} - \log Q n)^m \sim \Phi_m^{(d)}(\log Q n), \]

where \( \Phi_m^{(d)} \) is a 1-periodic function.

*Thus, \( M_n^{(d)} - \log Q n \) does not converge to a fixed limit law.*
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**Answer to above question is NO!**
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$q_{n,k} = \text{probability that a word satisfying GRGP has maximum value } k$. 
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With the same method as before:

**Proposition**

*Uniformly in a neighbourhood of 0,*

$$\mathbb{E}\left(e^{M_{n}^{(d)} t}\right) \sim \frac{P_0(e^t)\Omega_0(e^t)}{q^d\Omega_0(1)} n^{t/\log Q} \frac{\sum_{k} \Gamma(\log Q \rho_0 - t/\log Q + \chi_k) n^{-\chi_k}}{\sum_{k} \Gamma(\log Q \rho_0 + \chi_k) n^{-\chi_k}}.$$
Size of Block containing Largest Element

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For \( m \geq 1 \),

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where \( \Psi_m \) are 1-periodic functions.

Thus, \( N_n \) does not converge to a limit law.
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*Thus,* \( N_n \) *does not converge to a limit law.*

*Answer to above question is again NO!*
Summary

Seventh Cross-straight Conference on Combinatorics and Graph Theory:

$$\mathbb{E}(L_n^{(1)}) = \mathbb{E}(M_n^{(1)}) \sim \log Q n + \Phi_1^{(1)}(\log Q n).$$

I listed results for higher moments and limit laws as open problem.
Summary

Seventh Cross-straight Conference on Combinatorics and Graph Theory:

\[ \mathbb{E}(L_n^{(1)}) = \mathbb{E}(M_n^{(1)}) \sim \log_Q n + \Phi_1^{(1)}(\log_Q n). \]

I listed results for higher moments and limit laws as open problem.

These open problem were solved by F. & Javanian (2015):

<table>
<thead>
<tr>
<th>parameter</th>
<th>( m )-th central moments</th>
<th>limit law</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_n^{(d)} )</td>
<td>( \begin{cases} d = 1 : \text{periodic} \ d \geq 2 : \Theta((\log n)^{m/2}) \end{cases} )</td>
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