Distributional Results for the k-Robinson-Foulds Distance of Random Cayley Trees

(joint with Cheng-Kai Yeh and Mike Steel)

Michael Fuchs

Department of Mathematical Sciences
Chengchi University
Taipei, Taiwan

NATIONAL CHENGCHI UNIVERSITY

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Theorem (Penny & Steel; 1993)

As $n \to \infty$,

$$n - 3 - \frac{d_{RF}(\mathscr{T}_1, \mathscr{T}_2)}{2} \xrightarrow{d} \text{Poisson}(1/8).$$

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The RF-distance was recently defined for (unrooted and rooted) Cayley's trees in:

E. Khayatian, G. Valiente, L. Zhang (2024). The k-Robinson-Foulds measure for labeled trees, Journal of Comput. Biol., 31:4, 328–344.

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- Define:

$$N_e(u,k) := \{w \in C_u \ : \ d(w,u) \le k\};$$
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- Let $L_k(T)$ be the set of all k-local splits.
- For Cayley trees T_1 and T_2:

$$d_{k-RF}(T_1, T_2) := |L_k(T_1) \Delta L_k(T_2)|.$$

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Example:

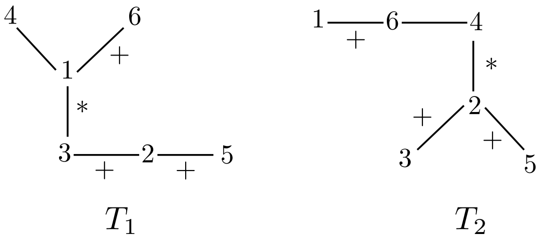
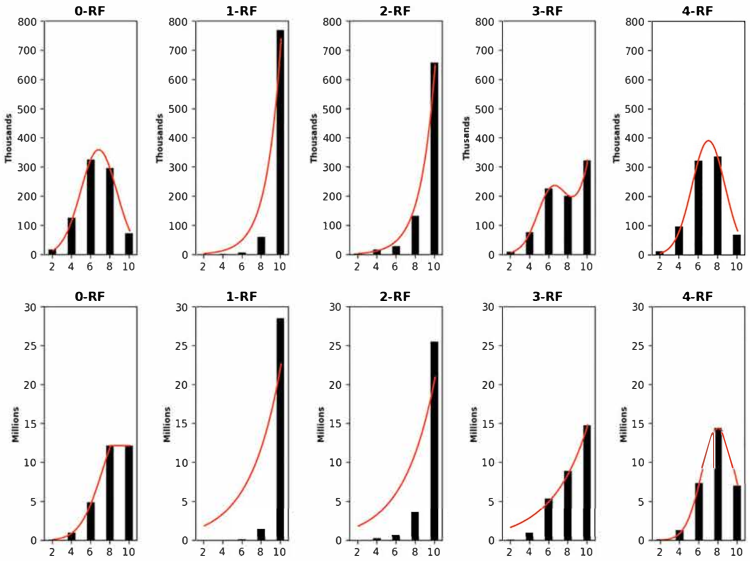


Figure: Shared k-local splits for $k=0$ (+) and $k=4$ (*).

Histogram for $n = 6$ (from Khayatian & Valiente & Zhang)



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(i) *For* $k=0$,

$$n-1-\frac{d_{0-RF}(\mathcal{T}_1,\mathcal{T}_2)}{2}\xrightarrow{d}\text{Poisson}(2).$$

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(ii) *For* $k = n-2$,

$$\frac{d_{(n-2)-RF}(\mathcal{T}_1, \mathcal{T}_2) - 2n(1-e^{-2})}{2\sqrt{(e^{-2} - 3e^{-4})n}} \xrightarrow{d} N(0,1).$$

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For the proof, we can equivalently work with $S_k(\mathscr{T}_1,\mathscr{T}_2)$.

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Thus the result follows from:

Proposition

As $n \to \infty$,

$$\frac{S'_n - ne^{-2}}{\sqrt{(e^{-2} - 3e^{-4})n}} \xrightarrow{d} N(0,1).$$

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Note:

$$\mathbb{E}(S'_n(S'_n-1)\cdots(S'_n-k+1))$$

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Corollary

(i) $\mathbb{E}(S'_n) \sim ne^{-2}$ *and* $\mathrm{Var}(S'_n) \sim (e^{-2} - 3e^{-4})n$.

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(i) $\mathbb{E}(S'_n) \sim ne^{-2}$ *and* $\mathrm{Var}(S'_n) \sim (e^{-2}-3e^{-4})n$.

(ii) *For* $m \geq 1$,

$$\mathbb{E}\left(\left(\frac{S'_n - ne^{-2}}{\sqrt{(e^{-2}-3e^{-4})n}}\right)^m\right) \sim \begin{cases} m!/(2^{m/2}(m/2)!), & \textit{if } m \textit{ is even;} \\ 0, & \textit{if } m \textit{ is odd.} \end{cases}$$

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Theorem (Gao & Wormald; 2004)

Let $s_n > -\mu_n^{-1}$ *and*

$$\sigma_n = \sqrt{\mu_n + \mu_n^2 s_n},$$

where $0 < \mu_n \to \infty$ *and* $\mu_n = o(\sigma_n^3)$. *Let* X_n *be a sequence of RVs with*

$$\mathbb{E}(X_n(X_n - 1)\cdots(X_n - k + 1)) \sim \mu_n^k e^{k^2 s_n/2}$$

uniformly for $c\mu_n/\sigma_n \le k \le c'\mu_n/\sigma_n$, *where* $c' > c > 0$.

Then, as $n \to \infty$,

$$\frac{X_n - \mu_n}{\sigma_n} \xrightarrow{d} N(0,1).$$

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$$P_n(y) = \frac{(2n-4)!}{n^{2n-4}}[z^{2n-4}](e^z + y - 1)^n.$$

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Now, use saddle point method.

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Proposition

The number of Cayley trees which contain a spanning forest F consisting of m trees equals:

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Thus, one can use the "dissociated case" of the Stein-Chen bound to prove the following result:

Proposition

We have,

$$d_{TV}(S_n, \mathrm{Poisson}(2(n-1)/n)) = O(1/n).$$

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On the other hand, the result for $k = n - 2$ remains unchanged.

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  E. Khayatian and L. Zhang. Simple k-RF Metrics for Comparison of Labeled DAGs, bioRxiv

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  a central limit theorem is conjectured for $d_{k-s-\mathrm{RF}}(\mathscr{T}_1, \mathscr{T}_2)$ for $1 \le k \le n-1$. This conjecture can also be proved with our tools (joint with Bernhard Gittenberger, TU Wien).

Reference

M. Fuchs and M. Steel. The asymptotic distribution of the k-Robinson-Foulds dissimilarity measure on labeled trees, Journal of Comput. Biol., in press.

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Thanks for the attention!