A Higher-dimensional Kurzweil Theorem for Formal Laurent Series over Finite Fields

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Abstract

In a recent paper, Kim and Nakada proved an analogue of Kurzweil's theorem for inhomogeneous Diophantine approximation of formal Laurent series over finite fields. Their proof used continued fraction theory and thus cannot be easily extended to simultaneous Diophantine approximation. In this note, we give another proof which works for simultaneous Diophantine approximation as well.

1 Introduction and Result

We start by fixing some notation which we are going to use throughout this work. First, let $\mathbb{F}_q$ denote the finite field with q elements. Moreover, denote by $\mathbb{F}_q[T]$ the polynomial ring and by

$$\mathbb{F}_q((T^{-1})) = \{f = a_nT^n + a_{n-1}T^{n-1} + \cdots \ : \ a_i \in \mathbb{F}_q, n \in \mathbb{Z}\}$$

the field of formal Laurent series.

For a formal Laurent series $f = a_nT^n + a_{n-1}T^{n-1} + \cdots$, we define its fractional part $\{f\}$ by

$$\{f\} = a_{-1}T^{-1} + a_{-2}T^{-2} + \cdots .$$

and its valuation by $|f| = q^{\deg f}$, where $\deg f$ is the generalized degree function. It is straightforward to prove that $|\cdot|$ satisfies the ultra-metric property, i.e., $|f-g| \le \max\{|f|, |g\}$ for all $f, g \in \mathbb{F}_q((T^{-1}))$ with equality whenever $|f| \neq |g|$. This property implies that balls, which we denote by

$$B(f; q^{-d}) = \{g \in \mathbb{F}_q((T^{-1})) \ : \ |g - f| < q^{-d}\},$$

are either disjoint or contained in each other.

Next, let

$$\mathbb{L} = \{f \in \mathbb{F}_q((T^{-1})) \ : \ |f| < 1\}.$$

Restricting the valuation to this set gives a compact topological group. Hence, there exists a unique, translation-invariant probability measure (the Haar measure) which we are going to denote by m.

In several recent papers, the following inhomogeneous Diophantine approximation problem was investigated: for $f, g \in \mathbb{L}$ consider

$$|\{Qf\} - g| < \frac{1}{q^{n+l_n}}, Q \in \mathbb{F}_q[T], \deg Q = n, \tag{1}$$

where l_n is a sequence of non-negative integers. One is interested in the number of solutions in Q of (1). Three situations have been studied: (D) f and g are both random; (S1) g is fixed; f is random; (S2) f is fixed; g is random. The first case is called the *double-metric* case and the other two cases are called *single-metric* cases.

We are going to recall some previous results concerning the number of solutions of (1). First, in all three cases, it follows immediately from the Borel-Cantelli lemma that the number of solutions is finite almost surely whenever $\sum_{n\geq 0} q^{-l_n}$ converges. Moreover, in the double-metric case and the single-metric case (S1) it was proved by Fuchs [2] and Ma and Su [7] that divergence of the latter series entails that the number of solutions is infinite almost surely. Interestingly, the same result does not hold for the single-metric case (S2). More precisely, for some functions f, the number of solutions remains finite almost surely even for sequences l_n for which $\sum_{n\geq 0} q^{-l_n} = \infty$. This then raises to question of characterizing those f where the convergence or divergence of $\sum_{n\geq 0} q^{-l_n}$ determines whether the number of solutions is finite or infinite almost surely.

To this end, we define the following set

$$W = \{f \in \mathbb{L} \ : \ \forall\, l_n \text{ with } \sum_{n=0}^{\infty} q^{-l_n} = \infty, \text{(1) has infinitely many solutions for almost all } g.\}$$

A characterization of this set was given in a recent paper by Kim and Nakada [4], their result being an analogue of Kurzweil's theorem from the real case. In order to state the result, we need a notation: $f \in \mathbb{L}$ is called *badly approximable* if there exists a $c \in \mathbb{N}$ such that for all $Q \in \mathbb{F}_q[T]$ with $\deg Q = n$, we have

$$|\{Qf\}| \geq \frac{1}{q^{n+c}}.$$

Then, Kim and Nakada proved the following result.

Theorem 1 (Kim and Nakada)**.** *We have,*

$$W = \{f \in \mathbb{L} \ : \ f \textit{ is badly approximable}\}.$$

As for the proof of the above result, Kim and Nakada used continued fraction theory. Hence, their proof is not easily extended to simultaneous Diophantine approximation. It is the purpose of this note to give another proof which works for simultaneous Diophantine approximation as well. Our new approach combines ideas of Kurzweil's original proof [6] and Kim and Nakada's approach from [4].

In order to state our result, we need further notation. Therefore, fix non-negative integers r and s. Then, we denote by $\mathbb{F}_q[T]^r$ the r-th fold Cartesian product of $\mathbb{F}_q[T]$ and by $\mathbb{F}_q((T^{-1}))^r$ the r-th dimensional vector space over $\mathbb{F}_q((T^{-1}))$. Throughout this work, vectors will always be row vectors and will be denoted by bold, lower-case letters.

Let $\mathbf{f} = (f_1, \ldots, f_r) \in \mathbb{F}_q((T^{-1}))^r$ be a vector. Then, we define its fractional part by

$$\{\mathbf{f}\} = (\{f_1\}, \ldots, \{f_r\})$$

and its valuation $\|\mathbf{f}\| = q^{\deg \mathbf{f}} = \max_{1\leq i\leq r} |f_i|$, where $\deg \mathbf{f} = \max_{1\leq i\leq r} \deg f_i$. Note that $\|\cdot\|$ again satisfies the ultra-metric property and balls

$$B(\mathbf{f}; q^{-d}) = \{\mathbf{g} \in \mathbb{F}_q((T^{-1}))^r \ : \ \|\mathbf{g} - \mathbf{f}\| < q^{-d}\}$$

are again either disjoint or contained in each other.

Finally, we let $\mathbb{L}^r$ denote the r-th fold Cartesian product of $\mathbb{L}$ which we equip with the product measure of $\mathbb{L}$ (also denoted by m). Note that due to Tychonov's theorem, $\mathbb{L}^r$ is again a compact topological group and hence the product measure is the unique Haar measure.

Now, we consider the following extension of (1): for $A \in \mathbb{L}^{r\times s}$ and $\mathbf{g} \in \mathbb{L}^s$ consider

$$\|\{\mathbf{q}A\} - \mathbf{g}\| < \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor + l_n}}, \ \mathbf{q} \in \mathbb{F}_q[T]^r, \ \deg \mathbf{q} = n, \tag{2}$$

where l_n is a sequences of non-negative integers. Again, one has three cases: (D) A and $\mathbf{g}$ are both random; (S1) $\mathbf{g}$ is fixed and A is random; (S2) A is fixed and $\mathbf{g}$ is random.

In this note, we are interested in case (S2). We mention in passing that similar results as in the one-dimensional case have been proved for the double-metric case and the single-metric case (S1) by Kristensen in [5]. So, the only case which has not been studied yet is (S2). In this case, we again have from the Borel-Cantelli lemma that if $\sum_{n\geq 0} q^{-l_n s}$ is convergent, then the number of solutions of (2) is finite almost surely. As for the other direction, we again define the set

$$W_{r,s} = \{A \in \mathbb{L}^{r\times s} \ : \ \forall\, l_n \text{ with } \sum_{n=0}^{\infty} q^{-l_n s} = \infty, \text{(2) has infinitely many solutions for almost all } \mathbf{g}.\}$$

We need the following notation: $A \in \mathbb{L}^{r\times s}$ is called *badly approximable* if there exists a $c \in \mathbb{N}$ such that for all $\mathbf{q} \in \mathbb{F}_q[T]^r$ with $\deg \mathbf{q} = n$, we have

$$\|\{\mathbf{q}A\}\| \geq \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor + c}}. \tag{3}$$

Then, our main result is the following extension of Theorem 1.

Theorem 2. *We have,*

$$W_{r,s} = \{A \in \mathbb{L}^{r\times s} \ : \ A \text{ is badly approximable}\}.$$

The structure of the paper is as follows: in the next section, we will collect a couple of results which are needed in the proof of Theorem 2. The proof of Theorem 2 is then presented in Section 3.

2 Some Preliminaries

Throughout this section, let $A \in \mathbb{L}^{r\times s}$ with

$$A = \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,s} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ f_{r,1} & f_{r,2} & \cdots & f_{r,s} \end{pmatrix}.$$

We first recall the higher-dimensional version of Dirichlet's theorem.

Theorem 3. *The following diophantine inequality*

$$\|\{\mathbf{q}A\}\| < \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor}}, \mathbf{q} \in \mathbb{F}_q[T]^r, \deg \mathbf{q} = n$$

has infinitely many solutions.

Proof. This is proved as in the real case. ▮

Next, we need the the following result.

Lemma 1. *If $A\mathbf{u}^{\intercal} \in \mathbb{F}_q[T]^r$ for some $\mathbf{u} \in \mathbb{F}_q[T]^s$ with $\mathbf{u} \neq \mathbf{0}$, then A is not badly approximable.*

Proof. Let $\mathbf{u} = (U_1, \dots, U_s)$ with $U_j \in \mathbb{F}_q[T]$ and assume w.l.o.g. that $U_s \neq 0$. From the assumption, we obtain that

$$A\mathbf{u}^\intercal = (V_1, \dots, V_r)^\intercal$$

with $V_i = \sum_{j=1}^{s} f_{i,j} U_j \in \mathbb{F}_q[T]$.

Next, denote by A' the matrix A with the last column removed. Then, by Dirichlet's theorem,

$$\|\{\mathbf{q}A'\}\| < q^{-\lfloor \frac{nr}{s-1} \rfloor}, \mathbf{q} \in \mathbb{F}_q[T]^r, \deg \mathbf{q} = n$$

has infinitely many solutions. The latter is equivalent to

$$\begin{aligned}
|Q_1 f_{1,1} + Q_2 f_{2,1} + \dots + Q_r f_{r,1} - P_1| &< q^{-\lfloor \frac{nr}{s-1} \rfloor}, \\
|Q_1 f_{1,2} + Q_2 f_{2,2} + \dots + Q_r f_{r,2} - P_2| &< q^{-\lfloor \frac{nr}{s-1} \rfloor}, \\
&\vdots \\
|Q_1 f_{1,s-1} + Q_2 f_{2,s-1} + \dots + Q_r f_{r,s-1} - P_{s-1}| &< q^{-\lfloor \frac{nr}{s-1} \rfloor}
\end{aligned}$$

has infinitely many solutions in $Q_1, \dots, Q_r, P_1, \dots, P_{s-1}$ with $\max_{1 \le i \le r} \deg Q_i = n$. Multiplying by U_s and setting $Q_i' = U_s Q_i, 1 \le i \le r$ and $P_j' = U_s P_j, 1 \le j \le s-1$ implies that

$$\begin{aligned}
|Q_1' f_{1,1} + Q_2' f_{2,1} + \dots + Q_r' f_{r,1} - P_1'| &< q^{-\lfloor \frac{n'r}{s-1} \rfloor - c_1}, \\
|Q_1' f_{1,2} + Q_2' f_{2,2} + \dots + Q_r' f_{r,2} - P_2'| &< q^{-\lfloor \frac{n'r}{s-1} \rfloor - c_1}, \\
&\vdots \\
|Q_1' f_{1,s-1} + Q_2' f_{2,s-1} + \dots + Q_r' f_{r,s-1} - P_{s-1}'| &< q^{-\lfloor \frac{n'r}{s-1} \rfloor - c_1}
\end{aligned} \tag{4}$$

has infinitely many solutions, where $\max_{1 \le i \le r} \deg Q_i' = n'$ and c_1 is a suitable constant.

Now, fix a solution of the latter system and observe that

$$\begin{aligned}
&U_s Q_1' f_{1,s} + U_s Q_2' f_{2,s} + \dots + U_s Q_r' f_{r,s} \\
&\quad = \sum_{i=1}^{r} (V_i - U_1 f_{i,1} - \dots - U_{s-1} f_{i,s-1}) Q_i' \\
&\quad = \sum_{i=1}^{r} V_i Q_i' - \sum_{j=1}^{s-1} U_j (Q_1' f_{1,j} + \dots + Q_r' f_{r,j} - P_j') - \sum_{j=1}^{s-1} U_j P_j'.
\end{aligned}$$

Rearranging yields

$$U_s \sum_{i=1}^{r} Q_i' f_{i,s} + \sum_{j=1}^{s-1} U_j P_j' - \sum_{i=1}^{r} V_i Q_i' = -\sum_{j=1}^{s-1} U_j (Q_1' f_{1,j} + \dots + Q_r' f_{r,j} - P_j')$$

Hence,

$$\left| U_s \sum_{i=1}^{r} Q_i' f_{i,s} + \sum_{j=1}^{s-1} U_j P_j' - \sum_{i=1}^{r} V_i Q_i' \right| \le \max_{1 \le j \le s-1} |U_j| |Q_1' f_{1,j} + \dots + Q_r' f_{r,j} - P_j'| < q^{-\lfloor \frac{n'r}{s-1} \rfloor - c_2},$$

where the last line follows from (4) and c_2 is a suitable constant. Dividing both sides by $|U_s|$ gives

$$\left| \sum_{i=1}^{r} Q_i' f_{i,s} + \frac{\sum_{j=1}^{s-1} U_j P_j' - \sum_{i=1}^{r} V_i Q_i'}{U_s} \right| < q^{-\lfloor \frac{n'r}{s-1} \rfloor - c_3},$$

where c_3 is a suitable constant. Note that $U_s|Q_i', 1 \leq i \leq r$ and $U_s|P_j', 1 \leq j \leq s-1$ and hence

$$T = \frac{\sum_{j=1}^{s-1} U_j P_j' - \sum_{i=1}^{r} V_i Q_i'}{U_s}$$

is a polynomial. Overall, we have proved that

$$|Q_1' f_{1,s} + \cdots + Q_r' f_{r,s} + T| < q^{-\lfloor \frac{n'r}{s-1} \rfloor - c_3}.$$

So, we can add this equation to (4) and the resulting system still has infinitely many solutions. This in turn yields that if we set $\mathbf{q}' = (Q_1', \ldots, Q_r')$ and $c_4 = \min\{c_1, c_3\}$, then

$$\|\{\mathbf{q}' A\}\| < q^{-\lfloor \frac{n'r}{s-1} \rfloor - c_4}, \mathbf{q}' \in \mathbb{F}_q[T]^r, \deg \mathbf{q}' = n \tag{5}$$

has infinitely many solutions.

The latter, however, implies that A is not badly approximable because otherwise (3) would hold which clearly contradicts (5). Hence, the proof is finished. ∎

Remark 1. In the real case, a matrix A is badly approximable if and only if $A^{\intercal}$ is badly approximable (see Theorem VIII in [1]). If the same is true for formal Laurent series as well (which we expect), then Lemma 1 would follow from this as a simple consequence.

For the final two results of this section, assume that A is badly approximable, i.e., (3) holds.

Lemma 2. *The set* $\{\{\mathbf{q}A\} : \mathbf{q} \in \mathbb{F}_q[T]^r\}$ *is dense in* $\mathbb{L}^s$.

Proof. Fix $n \in \mathbb{N}$ and $\mathbf{g} = (g_1, \ldots, g_s) \in \mathbb{L}^s$ with

$$g_j = g_1^{(j)} T^{-1} + g_2^{(j)} T^{-2} + \cdots .$$

We have to show that there exists a $\mathbf{q} \in \mathbb{F}_q[T]^r$ with

$$\|\{\mathbf{q}A\} - \mathbf{g}\| < q^{-n} \tag{6}$$

In order to do so, we reformulate (6) as a solvability problem for a system of linear equations. Therefore, let $\mathbf{q} = (Q_1, \ldots, Q_r)$ with

$$Q_i = a_0^{(i)} + a_1^{(i)} T + \cdots + a_N^{(i)} T^N$$

and for $1 \leq i \leq r$ and $1 \leq j \leq s$

$$f_{i,j} = f_1^{(i,j)} T^{-1} + f_2^{(i,j)} T^{-2} + \cdots .$$

Moreover,

$$\mathbf{u}_i = \begin{pmatrix} a_0^{(i)} \\ a_1^{(i)} \\ \vdots \\ a_N^{(i)} \end{pmatrix}^{\intercal}, \quad A_{i,j} = \begin{pmatrix} f_1^{(i,j)} & f_2^{(i,j)} & \cdots & f_n^{(i,j)} \\ f_2^{(i,j)} & f_3^{(i,j)} & \cdots & f_{n+1}^{(i,j)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{N+1}^{(i,j)} & f_{N+2}^{(i,j)} & \cdots & f_{N+n}^{(i,j)} \end{pmatrix}, \quad \mathbf{v}_j = \begin{pmatrix} g_1^{(j)} \\ g_2^{(j)} \\ \vdots \\ g_n^{(i,j)} \end{pmatrix}^{\intercal}$$

for $1 \leq i \leq r$ and $1 \leq j \leq s$. Finally, set

$$\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_r \end{pmatrix}^{\intercal}, \quad A' = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,s} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r,1} & A_{r,2} & \cdots & A_{r,s} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_s \end{pmatrix}^{\intercal}.$$

Then, (6) has a solution if and only if the system of linear equations $\mathbf{u}A' = \mathbf{v}$ has a solution $\mathbf{u}$.

In order to show that the latter system is solvable, it suffices to show that $\mathrm{rank}(A') = ns$ for N large enough. Assume that this is wrong. Then, there exist $\alpha_1, \dots, \alpha_{ns}$ not all 0 with

$$\alpha_1\left(f_1^{(1,1)}, \dots, f_{N+1}^{(1,1)}, f_1^{(2,1)}, \dots, f_{N+1}^{(2,1)}, \dots, f_1^{(r,1)}, \dots, f_{N+1}^{(r,1)}\right) \\ + \dots + \alpha_{ns}\left(f_n^{(1,s)}, \dots, f_{N+n}^{(1,s)}, f_n^{(2,s)}, \dots, f_{N+n}^{(2,s)}, \dots, f_n^{(r,s)}, \dots, f_{N+n}^{(r,s)}\right) = \mathbf{0}. \quad (7)$$

If we now set $\mathbf{u} = (U_1, \dots, U_s)$ with

$$\begin{aligned} U_1 &= \alpha_1 + \alpha_2 T + \dots + \alpha_n T^{n-1}, \\ U_2 &= \alpha_{n+1} + \alpha_{n+2} T + \dots + \alpha_{2n} T^{n-1}, \\ &\vdots \\ U_s &= \alpha_{n(s-1)+1} + \alpha_{n(s-1)+2} T + \dots + \alpha_{ns} T^{n-1}, \end{aligned}$$

then (7) can be reformulated as

$$|\{f_{i,1}U_1 + \dots + f_{i,s}U_s\}| < q^{-N-1}$$

for $1 \le i \le r$. This in turn gives that

$$\|A\mathbf{u}^{\intercal}\| < q^{-N-1}. \quad (8)$$

Now, since A is badly approximable, Lemma 1 implies that $\|A\mathbf{u}^{\intercal}\| > 0$. Consequently, since there are only finitely many possible choices of $\mathbf{u}$ (since n is fixed), (8) becomes wrong if N is large enough. This gives a contradiction and hence our result is proved. ▮

Lemma 3. *Let $E \subseteq \mathbb{L}^s$ and assume that E is invariant under the action $\cdot + \{\mathbf{q}A\}$ for all $\mathbf{q} \in \mathbb{F}_q[T]^r$. Then, $m(E) = 0$ or $m(E) = 1$.*

Proof. First, recall from the introduction that $\mathbb{L}^s$ is a compact topological group and m is its Haar measure.

Now, assume that $m(E) > 0$. We have to show that $m(E) = 1$. In order to do so, we use Lebesgues's density theorem for compact topological groups (see Remark 5 on page 268 in [3]): for all $\epsilon > 0$, there exists a $d \in \mathbb{Z}$ with

$$\int \left| \chi_E(\mathbf{g}) - \frac{m\left(E \cap B\left(\mathbf{g}; q^{-d}\right)\right)}{m\left(B\left(\mathbf{g}; q^{-d}\right)\right)} \right| \mathrm{d}m < \epsilon m(E),$$

where χ_E denotes the indicator function of E. The latter implies that

$$\int_E \left(1 - \frac{m\left(E \cap B\left(\mathbf{g}; q^{-d}\right)\right)}{m\left(B\left(\mathbf{g}; q^{-d}\right)\right)}\right) \mathrm{d}m < \epsilon m(E).$$

Hence, there exists a $\mathbf{g} \in \mathbb{L}^s$ with

$$1 - \frac{m\left(E \cap B\left(\mathbf{g}; q^{-d}\right)\right)}{m\left(B\left(\mathbf{g}; q^{-d}\right)\right)} < \epsilon$$

and consequently,

$$m\left(E \cap B\left(\mathbf{g}; q^{-d}\right)\right) > (1-\epsilon) m\left(B\left(\mathbf{g}; q^{-d}\right)\right).$$

Since E is invariant under the action $\cdot + \{\mathbf{q}A\}$ and m is translation-invariant, we obtain

$$m\left(E \cap \left(B\left(\mathbf{g}; q^{-d}\right) + \{\mathbf{q}A\}\right)\right) > (1-\epsilon) m\left(B\left(\mathbf{g}; q^{-d}\right) + \{\mathbf{q}A\}\right)$$

for all $\mathbf{q} \in \mathbb{F}_q[T]^r$. This together with Lemma 2 clearly implies that $m(E) > 1 - \epsilon$ and since this holds for all $\epsilon > 0$, we have $m(E) = 1$ as desired. ▮

3 Proof of the Main Result

In this section, we will prove Theorem 2. We will start with the case where A is badly approximable. For the next two results again assume that A satisfies (3).

Lemma 4. *Let* $\mathbf{g} \in \mathbb{L}^s$ *and* $d > 0$. *Then, the number of* $\mathbf{q} \in \mathbb{F}_q[T]^r$ *with* $\deg \mathbf{q} \leq N$ *such that* $\{\mathbf{q}A\} \in B(\mathbf{g}; q^{-d})$ *is at most* $\max\{q^{Nr+cs-ds}, 1\}$.

Proof. First, fix $\mathbf{q}, \mathbf{q}' \in \mathbb{F}_q[T]^r$ with $\deg \mathbf{q}, \deg \mathbf{q}' \leq N$. Then, since A is badly approximable, we have

$$\|\{\mathbf{q}A\} - \{\mathbf{q}'A\}\| = \|\{(\mathbf{q} - \mathbf{q}')A\}\| \geq q^{-\lfloor \frac{\deg(\mathbf{q}-\mathbf{q}')r}{s} \rfloor - c} \geq q^{-\lfloor \frac{Nr}{s} \rfloor - c}.$$

This means that the distance between any two points $\{\mathbf{q}A\}$ and $\{\mathbf{q}'A\}$ is at least $q^{-\lfloor \frac{Nr}{s} \rfloor - c}$.

Now, we consider two cases.

Case 1. If $q^{-\lfloor \frac{Nr}{s} \rfloor - c} \geq q^{-d}$, then there is at most one point in $B(\mathbf{g}; q^{-d})$.

Case 2. If $q^{-\lfloor \frac{Nr}{s} \rfloor - c} < q^{-d}$, then the number of points in $B(\mathbf{g}; q^{-d})$ is at most

$$\frac{\left(q^{-d}\right)^s}{\left(q^{-\lfloor \frac{Nr}{s} \rfloor - c}\right)^s} \leq q^{Nr+cs-ds}.$$

Hence, our claimed result is proved. ∎

Lemma 5. *Let* l_n *be a sequence with* $\sum_{n \geq 0} q^{-l_n s} = \infty$. *Then, for all* $k \geq 0$

$$m\left(\bigcup_{n=k}^{\infty} \bigcup_{\deg \mathbf{q}=n} B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l_n}\right)\right) > \frac{1}{q^{cs+1}}.$$

Proof. We first exclude the case $q = 2$ and $r = 1$.

Let $l'_n = \max\{l_n, c+1\}$. Then, $\sum_{n \geq 0} q^{-l'_n s} = \infty$. We will use proof by contradiction. Therefore, assume that the claim is wrong. Hence, there exists a $k_0 \geq 0$ such that for all $N \geq k_0$, we have

$$m\left(\bigcup_{n=k_0}^{N} \bigcup_{\deg \mathbf{q}=n} B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right)\right) \leq q^{-cs-1}. \tag{9}$$

Next, define the following set

$$L_N = \left\{ \deg \mathbf{q} = N \;:\; \{\mathbf{q}A\} \in \bigcup_{n=k_0}^{N} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right) \right.$$
$$\left. \setminus \bigcup_{n=k_0}^{N-1} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right) \right\}.$$

Our first goal is to estimate the cardinality of L_N. Therefore, set

$$\bigcup_{n=k_0}^{N-1} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right) = \bigcup_i B\left(\{\mathbf{q}'_i A\}; q^{-d_i}\right),$$

where the $B(\{\mathbf{q}'_i A\}; q^{-d_i})$ are disjoint for all i. Then, from (9),

$$q^{-cs-1} \geq m\left(\bigcup_{n=k_0}^{N-1} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l'_n}\right)\right) = m\left(\bigcup_i B\left(\{\mathbf{q}'_i A\}; q^{-d_i}\right)\right) = \sum_i q^{-d_i s}.$$

Using Lemma (4) implies that the number of $\mathbf{q}$ with $\deg \mathbf{q} \leq N$ such that $\{\mathbf{q}A\} \in \bigcup_i B(\{\mathbf{q}_i' A\}; q^{-d_i})$ is at most

$$\sum_i \max\left\{q^{Nr+cs-d_i s}, 1\right\} = \max\left\{q^{Nr+cs}\sum_i q^{-d_i s}, q^{Nr}\right\} = q^{Nr}.$$

Hence, the number of elements in L_N is at least

$$q^{(N+1)r} - q^{Nr} - q^{Nr} = q^{Nr}(q^r - 2) = dq^{Nr},$$

where $d > 0$ is a constant.

Next, we claim that

$$\bigcup_{\mathbf{q}\in L_N} B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{Nr}{s}\rfloor - l_N'}\right) \subseteq \bigcup_{n=k_0}^{N} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s}\rfloor - l_n'}\right) \setminus \bigcup_{n=k_0}^{N-1} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s}\rfloor - l_n'}\right). \tag{10}$$

In order to show this, fix a $\mathbf{q} \in L_N$ and assume that there exists a $\mathbf{q}'$ with $\deg \mathbf{q}' = n < N$ such that

$$B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{Nr}{s}\rfloor - l_N'}\right) \cap B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s}\rfloor - l_n'}\right) \neq \emptyset.$$

Since we know that $\{\mathbf{q}A\} \notin B(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s}\rfloor - l_n'})$, we obtain that

$$B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s}\rfloor - l_n'}\right) \subseteq B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{Nr}{s}\rfloor - l_N'}\right)$$

and hence $\{\mathbf{q}'A\} \in B(\{\mathbf{q}A\}; q^{-\lfloor \frac{Nr}{s}\rfloor - l_N'})$. The number of $\mathbf{q}$ with $\deg \mathbf{q} \leq N$ and $\{\mathbf{q}A\}$ belonging to the latter set is, however, at most

$$\max\left\{q^{Nr+cs-(\lfloor \frac{Nr}{s}\rfloor + l_N')s}, 1\right\} \leq \max\left\{q^{(c+1)s - l_N' s}, 1\right\} = 1.$$

This gives a contradiction and hence (10) is established.

Finally, we claim that the balls appearing on the left-hand side of (10) are pairwise disjoint. Therefore, consider $\mathbf{q}_1, \mathbf{q}_2 \in L_N$ with

$$B\left(\{\mathbf{q}_1 A\}; q^{-\lfloor \frac{Nr}{s}\rfloor - l_N'}\right) \cap B\left(\{\mathbf{q}_2 A\}; q^{-\lfloor \frac{Nr}{s}\rfloor - l_N'}\right) \neq \emptyset.$$

Thus, these two balls are equal and hence

$$\|\{\mathbf{q}_1 A\} - \{\mathbf{q}_2 A\}\| = \|\{(\mathbf{q}_1 - \mathbf{q}_2)A\}\| < q^{-\lfloor \frac{Nr}{s}\rfloor - l_N'}.$$

Now, as above, the ball $B(\mathbf{0}; q^{-\lfloor \frac{Nr}{s}\rfloor - l_N'})$ contains at most one point $\{\mathbf{q}A\}$ with $\deg \mathbf{q} \leq N$. Consequently, $\mathbf{q}_1 = \mathbf{q}_2$ and our claim is proved.

Now, from (10) and the latter claim, we obtain

$$\begin{aligned}
m&\left(\bigcup_{n=k_0}^{N} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s}\rfloor - l_n'}\right)\right) \\
&\geq m\left(\bigcup_{n=k_0}^{N-1} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s}\rfloor - l_n'}\right)\right) + m\left(\bigcup_{\mathbf{q}\in L_N} B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{Nr}{s}\rfloor - l_N'}\right)\right) \\
&\geq m\left(\bigcup_{n=k_0}^{N-1} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s}\rfloor - l_n'}\right)\right) + dq^{Nr}\left(q^{-\lfloor \frac{Nr}{s}\rfloor - l_N'}\right)^s \\
&\geq m\left(\bigcup_{n=k_0}^{N-1} \bigcup_{\deg \mathbf{q}'=n} B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{nr}{s}\rfloor - l_n'}\right)\right) + dq^{-l_N' s}.
\end{aligned}$$

Iterating yields

$$m\left(\bigcup_{n=k_0}^{N}\bigcup_{\deg\mathbf{q}'=n}B\left(\{\mathbf{q}'A\};q^{-\lfloor\frac{nr}{s}\rfloor-l'_n}\right)\right)\geq d\sum_{n=k_0}^{N}q^{-l'_ns}.$$

Since $\sum_{n\geq 0}q^{-l'_ns}=\infty$ this gives a contradiction when N is large enough.

Now, what is left is to consider the case $q=2$ and $r=1$. Here, we note that since $\sum_{n\geq 0}q^{-l'_ns}=\infty$, we have either $\sum_{n\geq 0}q^{-l'_{2n}s}=\infty$ or $\sum_{n\geq 0}q^{-l'_{2n+1}s}=\infty$. W.l.o.g. assume that the first case holds. Then, the same proof as above can be used with the only difference that instead of L_N, we consider

$$L_{2N}=\left\{\deg\mathbf{q}=2N\ :\ \{\mathbf{q}A\}\in\bigcup_{n=k_0}^{2N}\bigcup_{\deg\mathbf{q}'=n}B\left(\{\mathbf{q}'A\};q^{-\lfloor\frac{nr}{s}\rfloor-l'_n}\right)\right.$$
$$\left.\setminus\bigcup_{n=k_0}^{2N-2}\bigcup_{\deg\mathbf{q}'=n}B\left(\{\mathbf{q}'A\};q^{-\lfloor\frac{nr}{s}\rfloor-l'_n}\right)\right\}.$$

Details are straightforward and we leave them to the reader. ∎

Now, we can prove one half of Theorem 2.

Proposition 1. *Let $A\in\mathbb{L}^{r\times s}$ be badly approximable. Then, for all sequences l_n with $\sum_{n\geq 0}q^{-l_ns}=\infty$, we have that (2) has infinitely many solutions for almost all $\mathbf{g}\in\mathbb{L}^s$.*

Proof. Consider

$$E=\bigcap_{k=0}^{\infty}\bigcup_{n=k}^{\infty}\bigcup_{\deg\mathbf{q}=n}B\left(\{\mathbf{q}A\};q^{-\lfloor\frac{nr}{s}\rfloor-l_n}\right).$$

Then, we have for all $\mathbf{g}\in\mathbb{L}^s$ that $\mathbf{g}\in E$ if and only if (2) has infinitely many solutions. Moreover, Lemma 5 implies that $m(E)>0$. Since E is invariant under the action $\cdot+\{\mathbf{q}A\}$ for all $\mathbf{q}\in\mathbb{F}_q[T]^r$, the latter and Lemma 3 yields $m(E)=1$ which is the desired result. ∎

In order to conclude the proof of Theorem 2 what is left is to consider the case where A is not badly approximable.

Proposition 2. *Let $A\in\mathbb{L}^{r\times s}$ be not badly approximable. Then, there exists a sequence l_n with $\sum_{n\geq 0}q^{-l_ns}=\infty$ but (2) has only finitely many solutions for almost all $\mathbf{g}\in\mathbb{L}^s$.*

Proof. First, since A is not badly approximable, there exists a sequence $\mathbf{q}_i=(Q_1^{(i)},\dots,Q_r^{(i)})\in\mathbb{F}_q[T]^r$ with $\deg\mathbf{q}_i=n_i$ and n_i increasing such that

$$\|\{\mathbf{q}_iA\}\|<q^{-\lfloor\frac{(n_i+i)r}{s}\rfloor-i}.$$

Now, define $t_0=0$ and $t_i=n_i+i$ for all i. Moreover, for n with $t_{i-1}\leq n<t_i$ set

$$l_n=\left\lfloor\frac{(t_i-n)r}{s}\right\rfloor.$$

Note that l_n is a sequence with

$$\sum_{n\geq 0}q^{-l_ns}\geq\sum_{i=1}^{\infty}q^{-l_{t_i-1}s}\geq\sum_{i=1}^{\infty}q^{-\lfloor\frac{r}{s}\rfloor s}\geq\sum_{i=1}^{\infty}q^{-r}=\infty.$$

Next, assume w.l.o.g. that $q^{n_i}=\|\mathbf{q}_i\|=|Q_1^{(i)}|$. We claim that

$$\bigcup_{t_{i-1}\leq n<t_i}\bigcup_{\deg\mathbf{q}=n}B\left(\{\mathbf{q}A\};q^{-\lfloor\frac{nr}{s}\rfloor-l_n}\right)\subseteq\bigcup B\left(\{\mathbf{q}'A\};q^{-\lfloor\frac{t_ir}{s}\rfloor+2}\right),$$

where the second union runs over all $\mathbf{q}' = (Q'_1, \dots, Q'_r)$ with

$$|Q'_1| \leq q^{n_i - 1}, |Q'_2| \leq q^{t_i - 1}, \dots, |Q'_r| \leq q^{t_i - 1}.$$

In order to show this, fix $\mathbf{q} = (Q_1, \dots, Q_r)$ with $t_{i-1} \leq \deg \mathbf{q} = n < t_i$. Using division with remainder gives a $P \in \mathbb{F}_q[T]$ with $|Q_1 + PQ_1^{(i)}| \leq q^{n_i - 1}$. Note that $|P| \leq q^{t_i - 1 - n_i}$. Now set

$$\mathbf{q}' = (Q_1 + PQ_1^{(i)}, \dots, Q_r + PQ_r^{(i)}).$$

Then,

$$\|\{\mathbf{q}A\} - \{\mathbf{q}'A\}\| \leq |P| \|\{\mathbf{q}_i A\}\| < q^{t_i - 1 - n_i - \lfloor \frac{t_i r}{s} \rfloor - i} = q^{-\lfloor \frac{t_i r}{s} \rfloor - 1}.$$

Also, note that

$$q^{-\lfloor \frac{nr}{s} \rfloor - l_n} = q^{-\lfloor \frac{nr}{s} \rfloor - \lfloor \frac{(t_i - n)r}{s} \rfloor} < q^{-\frac{nr}{s} - \frac{(t_i - n)r}{s} + 2} \leq q^{-\lfloor \frac{t_i r}{s} \rfloor + 2}.$$

Consequently,

$$B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l_n}\right) \subseteq B\left(\{\mathbf{q}'A\}; q^{-\lfloor \frac{t_i r}{s} \rfloor + 2}\right)$$

which proves the claim.

In order to conclude the proof, observe that the claim implies

$$m\left(\bigcup_{t_{i-1} \leq n < t_i} \bigcup_{\deg \mathbf{q} = n} B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l_n}\right)\right) \leq q^{(-\lfloor \frac{t_i r}{s} \rfloor + 2)s} q^{n_i + t_i(r-1)} < q^{3s - i}.$$

Hence,

$$\sum_{i=1}^{\infty} m\left(\bigcup_{t_{i-1} \leq n < t_i} \bigcup_{\deg \mathbf{q} = n} B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l_n}\right)\right) \leq \sum_{i=1}^{\infty} q^{3s - i} < \infty.$$

The Borel-Cantelli lemma now implies that for almost all $\mathbf{g} \in \mathbb{L}^s$

$$\mathbf{g} \in \bigcup_{t_{i-1} \leq n < t_i} \bigcup_{\deg \mathbf{q} = n} B\left(\{\mathbf{q}A\}; q^{-\lfloor \frac{nr}{s} \rfloor - l_n}\right)$$

for only finitely many n which proves the desired result. ∎

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