On the number of subtrees on the fringe of random trees

(partly joint with Huilan Chang)

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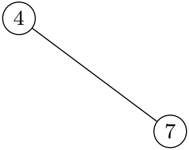

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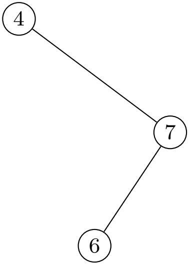

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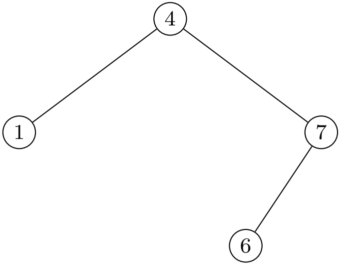

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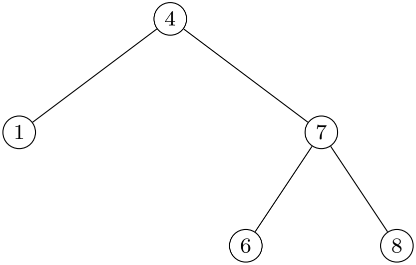

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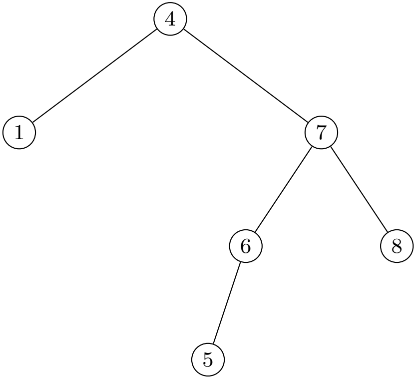

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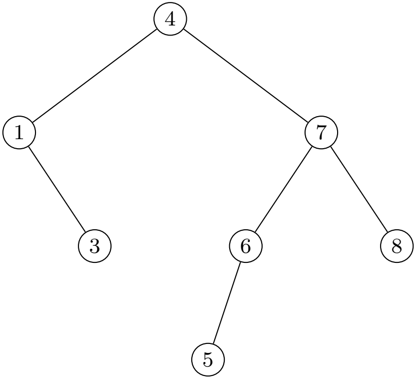

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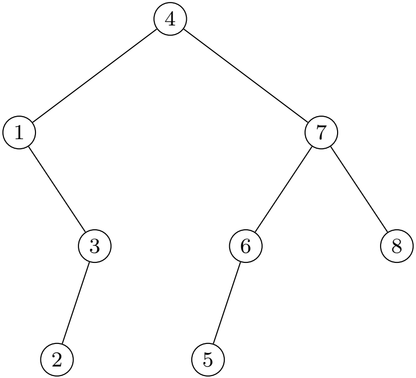

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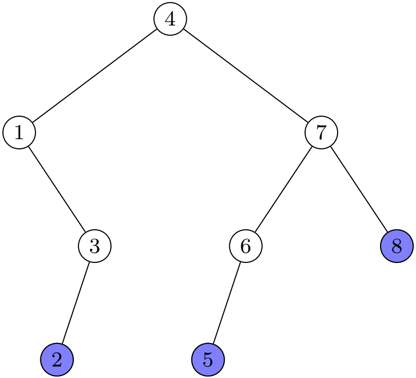

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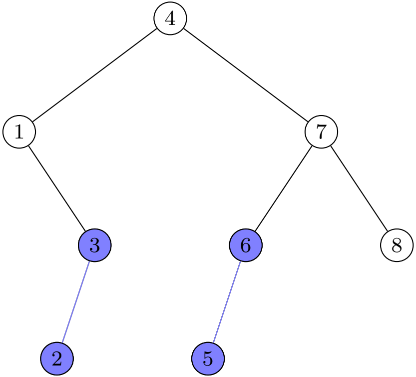

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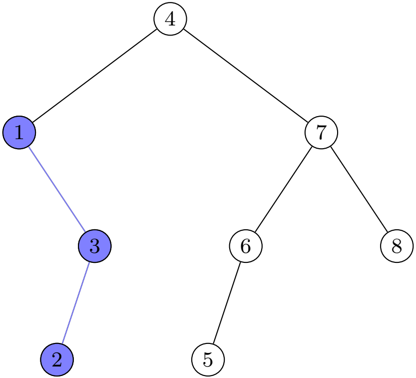

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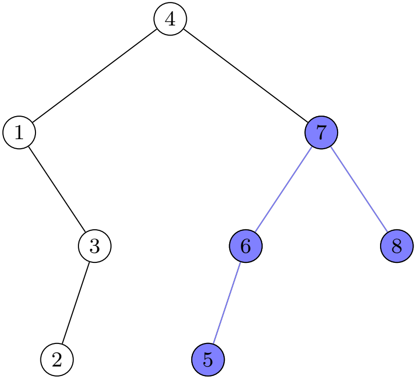

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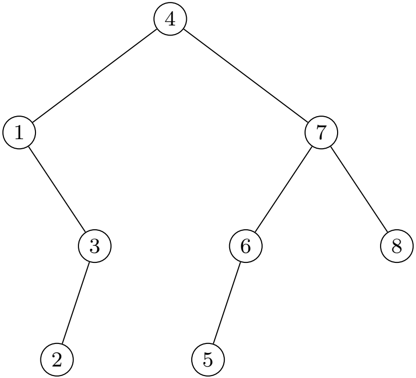

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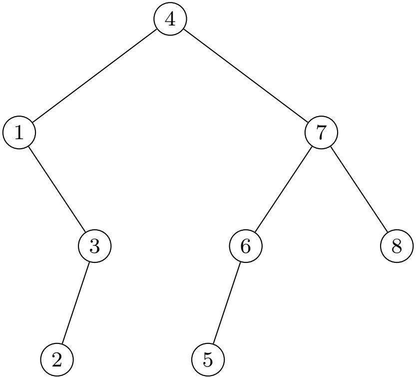

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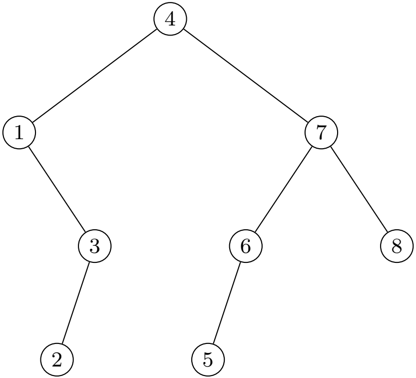

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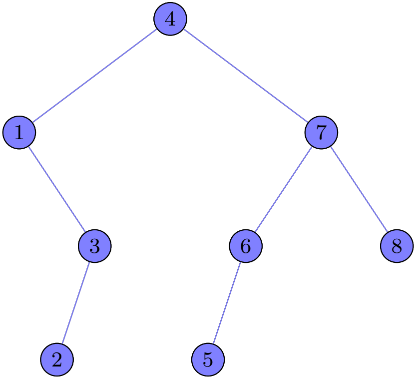

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Mean value and variance

$X_{n,k}$ satisfies

$$X_{n,k} \stackrel{d}{=} X_{I_n,k} + X^*_{n-1-I_n,k},$$

where $X_{k,k} = 1$, $X_{I_n,k}$ and $X^*_{n-1-I_n,k}$ are conditionally independent given I_n, and $I_n = \text{Unif}\{0, \ldots, n-1\}$.

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This yields

$$\mu_{n,k} := \mathbf{E}(X_{n,k}) = \frac{2(n+1)}{(k+1)(k+2)}, \qquad (n > k),$$

and

$$\sigma^2_{n,k} := \mathrm{Var}(X_{n,k}) = \frac{2k(4k^2+5k-3)(n+1)}{(k+1)(k+2)^2(2k+1)(2k+3)}$$

for $n > 2k+1$.

Some previous results

- Aldous (1991): Weak law of large numbers

$$\frac{X_{n,k}}{\mu_{n,k}} \longrightarrow 1 \qquad \text{in probability.}$$

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$\longrightarrow$ All the above results are for fixed k.

Results for $k = k_n$

Theorem (Feng, Mahmoud, Panholzer (2008))

(i) *(Normal range) Let* $k = o(\sqrt{n})$ *and* $k \to \infty$ *as* $n \to \infty$. *Then,*

$$\frac{X_{n,k} - \mu_{n,k}}{\sqrt{2n/k^2}} \xrightarrow{d} \mathcal{N}(0,1).$$

(ii) *(Poisson range) Let* $k \sim c\sqrt{n}$ *as* $n \to \infty$. *Then,*

$$X_{n,k} \xrightarrow{d} \mathrm{Poisson}(2c^{-2}).$$

(iii) *(Degenerate range) Let* $k < n$ *and* $\sqrt{n} = o(k)$ *as* $n \to \infty$. *Then,*

$$X_{n,k} \xrightarrow{L_1} 0.$$

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- The methods for proving phase change results might be applicable to other parameters which are expected to exhibit the same phase change behavior as well.
- $X_{n,k}$ is related to parameters arising in genetics.

Yule generated random genealogical trees

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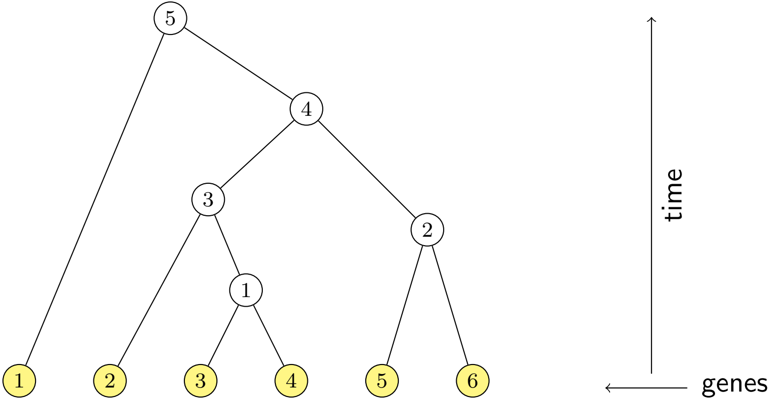

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Random model:

At every time point, two yellow nodes uniformly coalescent.

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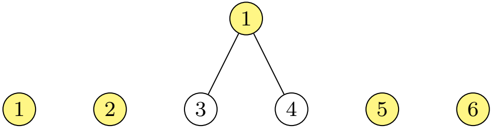

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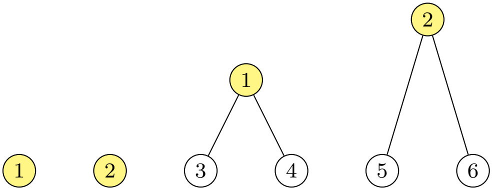

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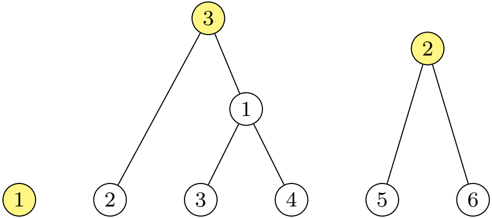

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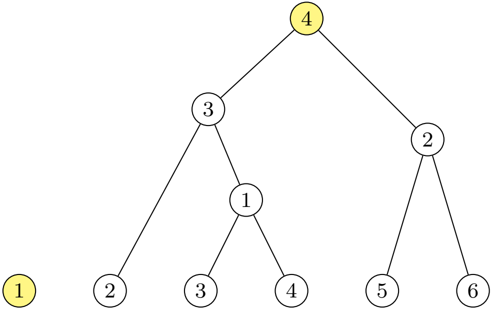

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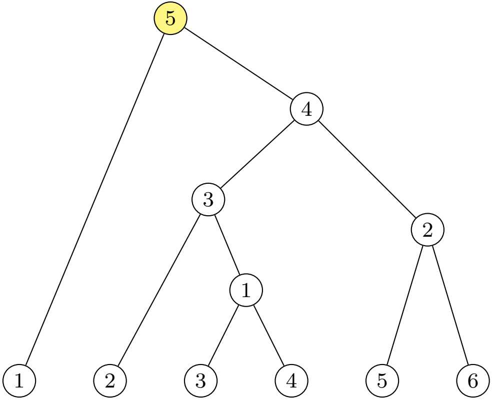

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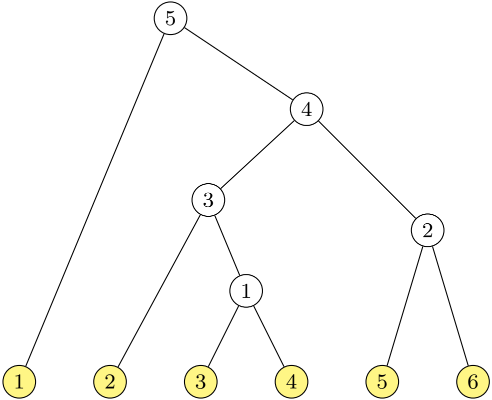

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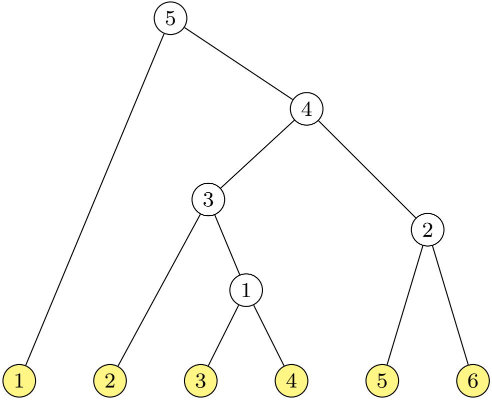

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Same model as random binary search tree model!

Shape parameters of genealogical trees

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- Nodes with minimal clade size k (Blum and François (2005)):

  If $k \geq 3$, then they are internal nodes with induced subtree of size $k-1$ and either an empty right subtree or empty left subtree.

Counting pattern in random binary search trees

Consider $X_{n,k}$ with

$$X_{n,k} \stackrel{d}{=} X_{I_n,k} + X^*_{n-1-I_n,k},$$

where $X_{k,k} = \text{Bernoulli}(p_k)$, $X_{I_n,k}$ and $X^*_{n-1-I_n,k}$ are conditionally independent given I_n, and $I_n = \text{Unif}\{0, \ldots, n-1\}$.

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Then,

p_k	shape parameter
1	$\#$ of $k+1$-pronged nodes
$2/k$	$\#$ of nodes with minimal clade size $k+1$
$2^{k-1}/k!$	$\#$ of $k+1$ caterpillars

Underlying recurrence and solution

All (centered or non-centered) moments satisfy

$$a_{n,k} = \frac{2}{n} \sum_{j=0}^{n-1} a_{j,k} + b_{n,k},$$

where $a_{k,k}$ is given and $a_{n,k} = 0$ for $n < k$.

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We have

$$a_{n,k} = \frac{2(n+1)}{(k+1)(k+2)} a_{k,k} + 2(n+1) \sum_{k<j<n} \frac{b_{j,k}}{(j+1)(j+2)} + b_{n,k},$$

where $n > k$.

Mean value and variance

We have

$$\mathbf{E}(X_{n,k}) = \frac{2(n+1)}{(k+1)(k+2)}p_k, \qquad (n > k),$$

and

$$\mathrm{Var}(X_{n,k}) = \frac{2p_k(4k^3+16k^2+19k+6-(11k^2+22k+6)p_k)(n+1)}{(k+1)(k+2)^2(2k+1)(2k+3)}$$

for $n > 2k+1$.

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Note that

$$\mathbf{E}(X_{n,k}) \sim \mathrm{Var}(X_{n,k}) \sim \frac{2p_k}{k^2}n$$

for $n > 2k+1$ and $k \to \infty$ as $n \to \infty$.

Higher moments

Denote by

$$A_{n,k}^{(m)} := \mathbf{E}(X_{n,k} - \mathbf{E}(X_{n,k}))^m.$$

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Then,

$$A_{n,k}^{(m)} = \frac{2}{n}\sum_{j=0}^{n-1} A_{j,k}^{(m)} + B_{n,k}^{(m)},$$

where

$$B_{n,k}^{(m)} := \sum_{\substack{i_1+i_2+i_3=m \\ 0\le i_1,i_2<m}} \binom{m}{i_1, i_2, i_3} \frac{1}{n}\sum_{j=0}^{n-1} A_{j,k}^{(i_1)} A_{n-1-j,k}^{(i_2)} \Delta_{n,j,k}^{i_3}$$

and

$$\Delta_{n,j,k} = \mathbf{E}(X_{j,k}) + \mathbf{E}(X_{n-1-j,k}) - \mathbf{E}(X_{n,k}).$$

Normal range

Proposition

Uniformly for $n, k, m \geq 1$ *and* $n > k$

$$A_{n,k}^{(m)} = \mathcal{O}\left(\max\left\{\frac{2p_k n}{k^2}, \left(\frac{2p_k n}{k^2}\right)^{m/2}\right\}\right).$$

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Proposition

For $\mathbf{E}(X_{n,k}) \to \infty$ as $n \to \infty$,

$$A_{n,k}^{(2m-1)} = o\left(\left(\frac{2p_k n}{k^2}\right)^{m-1/2}\right), \qquad A_{n,k}^{(2m)} \sim g_m \left(\frac{2p_k n}{k^2}\right)^m,$$

where

$$g_m = (2m)!/(2^m m!).$$

Poisson range

Consider

$$\bar{A}_{n,k}^{(m)} = \mathbf{E}(X_{n,k}(X_{n,k}-1)\cdots(X_{n,k}-m+1)).$$

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Then, similarly as before:

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(i) *Uniformly for* $n, k, m \geq 1$ *and* $n > k$

$$\bar{A}_{n,k}^{(m)} = \mathcal{O}\left(\max\left\{\frac{2p_k n}{k^2}, \left(\frac{2p_k n}{k^2}\right)^m\right\}\right).$$

(ii) *For* $\mathbf{E}(X_{n,k}) \to c$ *and* $k < n$ *as* $n \to \infty$,

$$\bar{A}_{n,k}^{(m)} \longrightarrow c^m.$$

The phase change

Theorem

(i) *(Normal range) Let* $\mathbf{E}(X_{n,k}) \to \infty$ *and* $k \to \infty$ *as* $n \to \infty$. *Then,*

$$\frac{X_{n,k} - \mathbf{E}(X_{n,k})}{\sqrt{2p_k n/k^2}} \xrightarrow{d} \mathcal{N}(0,1).$$

(ii) *(Poisson range) Let* $\mathbf{E}(X_{n,k}) \to c > 0$ *and* $k < n$ *as* $n \to \infty$. *Then,*

$$X_{n,k} \xrightarrow{d} \mathrm{Poisson}(c).$$

(iii) *(Degenerate range) Let* $\mathbf{E}(X_{n,k}) \to 0$ *as* $n \to \infty$. *Then,*

$$X_{n,k} \xrightarrow{L_1} 0.$$

A comparison of the phase change

For k-caterpillars, we have

$$\mathbf{E}(X_{n,k}) = \frac{2^{k-1}n}{(k+2)!}.$$

Note that either

$$\mathbf{E}(X_{n,k}) \to \infty \qquad \text{or} \qquad \mathbf{E}(X_{n,k}) \to 0.$$

So, there is no Poisson range.

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So, there is no Poisson range.

shape parameter	location	phase change
k-pronged nodes	$\sqrt{n}$	normal - poisson - degenerate
minimal clade size k	$\sqrt[3]{n}$	normal - poisson - degenerate
k-caterpillars	$\ln n/(\ln \ln n)$	normal - degenerate

Refined results (for # of subtrees)

Define

$$\phi_{n,k}(y) = e^{-\sigma_{n,k}^2 y^2/2} \mathbf{E}\left(e^{(X_{n,k}-\mu_{n,k})y}\right).$$

and

$$\phi_{n,k}^{(m)} = \left.\frac{d^m \phi_{n,k}(y)}{dy^m}\right|_{y=0}.$$

Refined results (for # of subtrees)

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and

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Proposition

Uniformly for $n, k \geq 1$ and $m \geq 0$

$$|\phi_{n,k}^{(m)}| \leq m! A^m \max\left\{\frac{n}{k^2}, \left(\frac{n}{k^2}\right)^{m/3}\right\}$$

for a suitable constant A.

Berry-Esseen bound and LLT for the normal range

Theorem (Rate of convergency)

For $1 \leq k = o(\sqrt{n})$ *as* $n \to \infty$,

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{X_{n,k} - \mu_{n,k}}{\sigma_{n,k}} < x \right) - \Phi(x) \right| = \mathcal{O}\left(\frac{k}{\sqrt{n}} \right).$$

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Theorem (LLT)

For $1 \le k = o(\sqrt{n})$ *as* $n \to \infty$,

$$P(X_{n,k} = \lfloor \mu_{n,k} + x\sigma_{n,k} \rfloor) = \frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma_{n,k}} \left(1 + \mathcal{O}\left((1+|x|^3) \frac{k}{\sqrt{n}} \right) \right),$$

uniformly in $x = o(n^{1/6}/k^{1/3})$.

Define

$$\bar{\phi}_{n,k}(y) = e^{-\mu_{n,k}(y-1)}\mathbf{E}\left(y^{X_{n,k}}\right).$$

and

$$\phi_{n,k}^{(m)} = \left.\frac{d^m\bar{\phi}_{n,k}(y)}{dy^m}\right|_{y=1}.$$

LLT for the Poisson range

Define

$$\bar{\phi}_{n,k}(y) = e^{-\mu_{n,k}(y-1)}\mathbf{E}\left(y^{X_{n,k}}\right).$$

and

$$\phi_{n,k}^{(m)} = \left.\frac{d^m\bar{\phi}_{n,k}(y)}{dy^m}\right|_{y=1}.$$

Proposition

Uniformly for $n > k$ and $m \geq 0$

$$|\bar{\phi}_{n,k}^{(m)}| \leq m!A^m\left(\frac{n}{k^3}\right)^{m/2}$$

for a suitable constant A.

Poisson approximation

Theorem (LLT)

For $k < n$ and $n \to \infty$,

$$P(X_{n,k} = l) = e^{-\mu_{n,k}} \frac{(\mu_{n,k})^l}{l!} + \mathcal{O}\left(\frac{n}{k^3}\right)$$

uniformly in l.

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uniformly in l.

Theorem (Poisson approximation)

Let $k < n$ *and* $k \to \infty$ *as* $n \to \infty$. *Then,*

$$d_{TV}(X_{n,k}, \mathrm{Poisson}(\mu_{n,k})) \longrightarrow 0.$$

Remark: A rate can be given as well.

Other types of random trees

- Random recursive trees

  Non-plane, labelled trees with every label sequence from the root to a leave increasing; random model is the uniform model.

  Methods works as well (with minor modifications) and similar results can be proved.

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- Random recursive trees

  Non-plane, labelled trees with every label sequence from the root to a leave increasing; random model is the uniform model.

  Methods works as well (with minor modifications) and similar results can be proved.

- Plane-oriented recursive trees (PORTs)

  Plane, labelled trees with every label sequence from the root to a leave increasing; random model is the uniform model.

  Method works as well, but details more involved.

Mean value and variance of PORTs

We have,

$$\mu_{n,k} := \mathbf{E}(X_{n,k}) = \frac{2n-1}{4k^2-1}, \qquad (n > k).$$

Mean value and variance of PORTs

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$$\mu_{n,k} := \mathbf{E}(X_{n,k}) = \frac{2n-1}{4k^2-1}, \qquad (n > k).$$

Moreover, for fixed k as $n \to \infty$,

$$\mathrm{Var}(X_{n,k}) \sim c_k n,$$

where

$$c_k = \frac{8k^2-4k-8}{(4k^2-1)^2} - \frac{((2k-3)!!)^2}{((k-1)!)^2 4^{k-1} k(2k+1)},$$

and, for $k < n$ and $k \to \infty$ as $n \to \infty$,

$$\mathbf{E}(X_{n,k}) \sim \mathrm{Var}(X_{n,k}) \sim \frac{n}{2k^2}.$$

The phase change

Theorem

(i) *(Normal range) Let* $k = o(\sqrt{n})$ *and* $k \to \infty$ *as* $n \to \infty$. *Then,*

$$\frac{X_{n,k} - \mu_{n,k}}{\sqrt{n/(2k^2)}} \xrightarrow{d} \mathcal{N}(0,1).$$

(ii) *(Poisson range) Let* $k \sim c\sqrt{n}$ *as* $n \to \infty$. *Then,*

$$X_{n,k} \xrightarrow{d} \mathrm{Poisson}((2c^2)^{-1}).$$

(iii) *(Degenerate range) Let* $k < n$ *and* $\sqrt{n} = o(k)$ *as* $n \to \infty$. *Then,*

$$X_{n,k} \xrightarrow{L_1} 0.$$

More results and future research

- Parameters of genealogical trees under different random models

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  Very simple classes of increasing trees and more general classes of increasing trees (polynomial varieties, mobile trees, etc.)

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- Parameters of genealogical trees under different random models
- Universality of the phase change for the number of subtrees

  Very simple classes of increasing trees and more general classes of increasing trees (polynomial varieties, mobile trees, etc.)
- Phase change results for the number of nodes with out-degree k

  Important in computer science.

  A phase change from normal to degenerate is expected (no Poisson range).